the Cosmic Web:

Lecture 2b:
Lagrangian Dynamics,
Multistream Structure & Caustic Skeleton

Rien van de Weijgaert
Excosm Cosmology Summerschool, Haapsalu, July 2025

Cosmic Web

Dynamics & Formation: Program

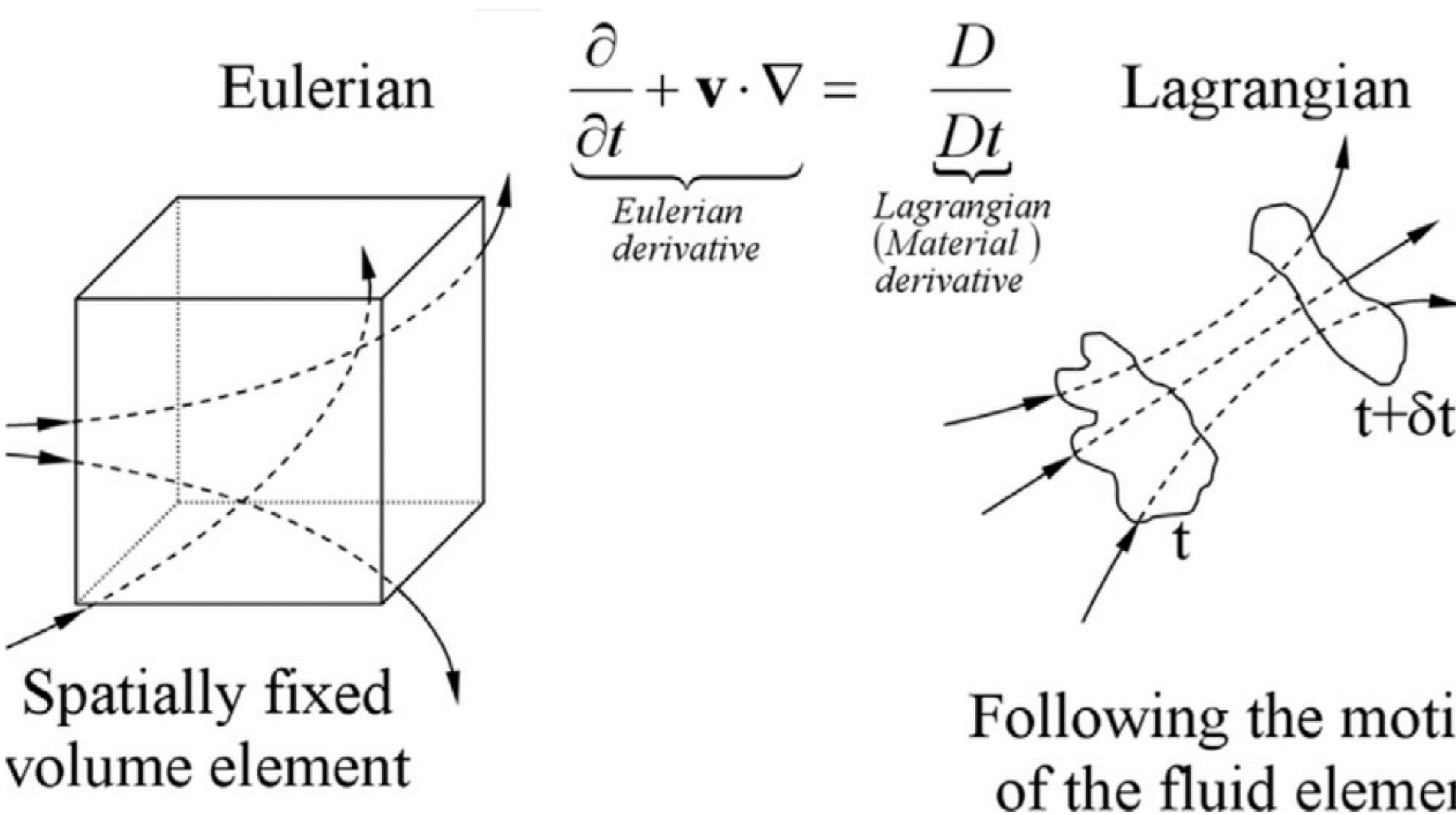
Cosmic Web – Formation & Dynamics

- Forces & Strains - Observational Manifestations
- the Mechanism - Gravitational Instability
- Anisotropic Collapse - Formation of filaments and walls
- Weaving the Web - Connection Clusters, Filaments and Walls
- Dynamical Inventory - Forces & Tides in the Cosmic Web
- Phase Space Dynamics - Phase Space & Multistream structure
- Lagrangian Dynamics - Zeldovich formalism
- Hierarchical Formation - from small to the Megaparsec Cosmic Web
- Anisotropy & Hierarchy - the Adhesion formalism
- Caustic Skeleton - analytical formalism cosmic web

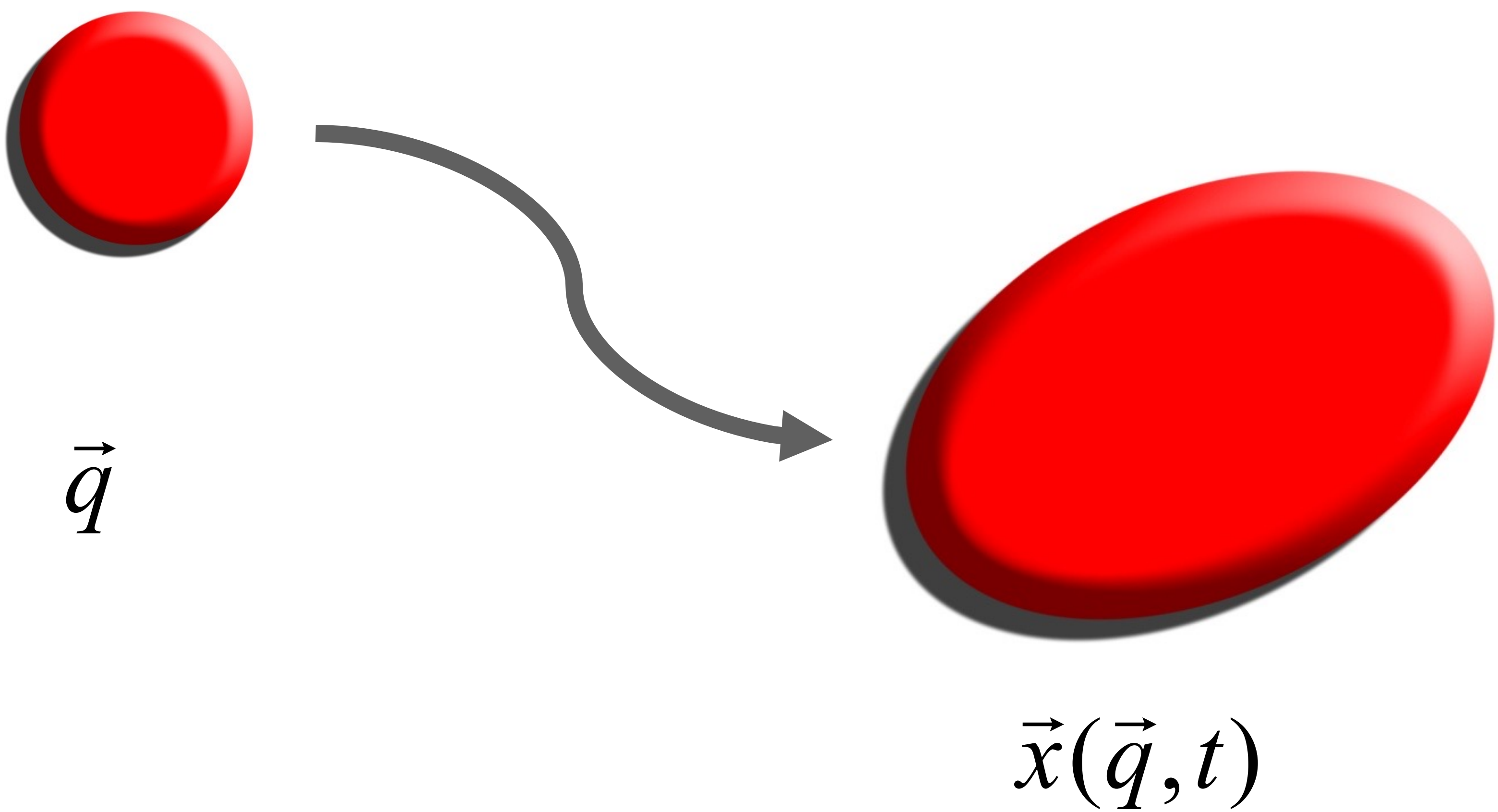
Lagrangian View

Structure Formation

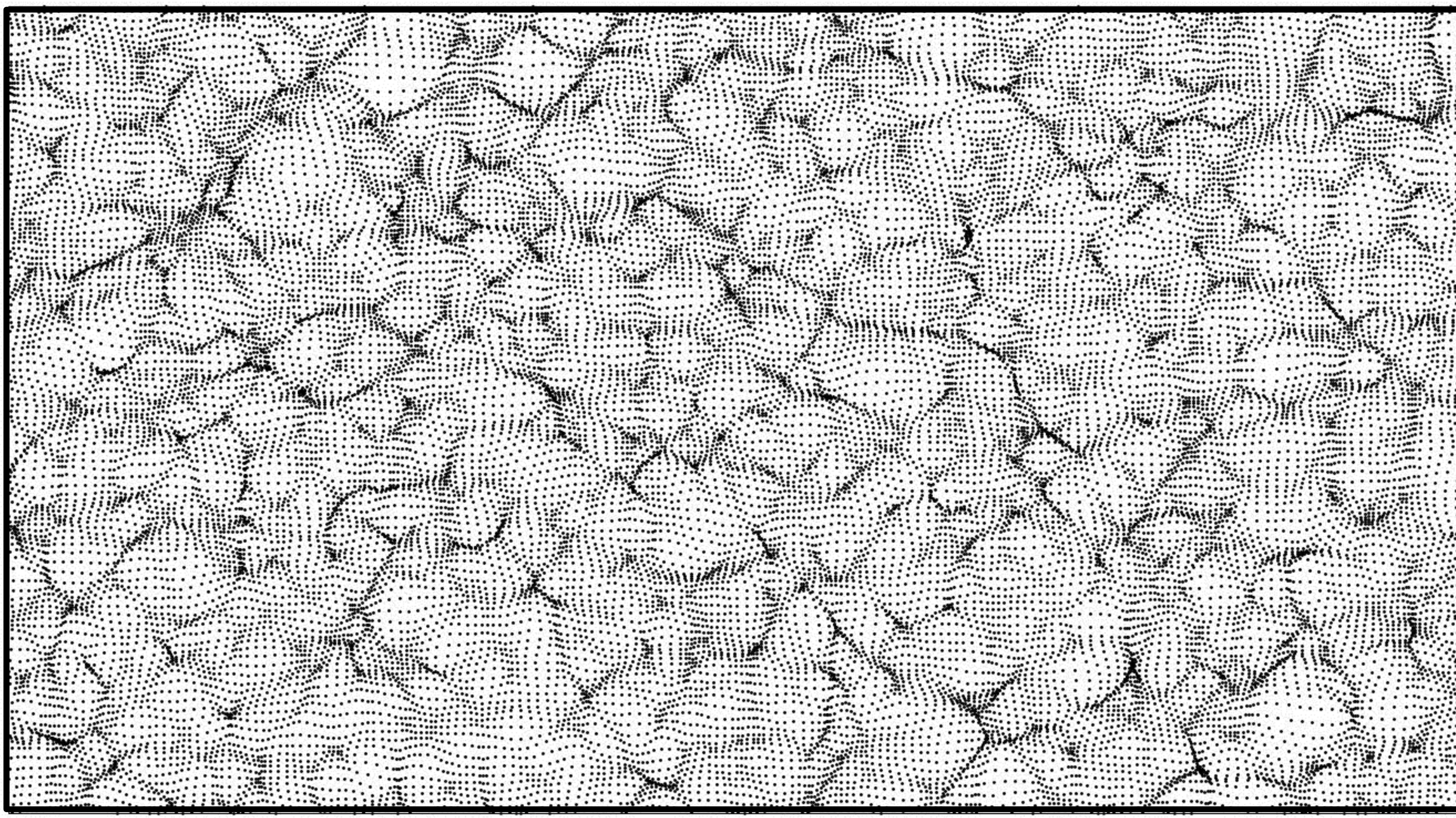
Eulerian - Lagrangian



Lagrangian - Eulerian



Simulation – Discrete Particles



Simulation – Mass Elements

Cosmic Web

Phase-Space Dynamics
&
Multistream Regions

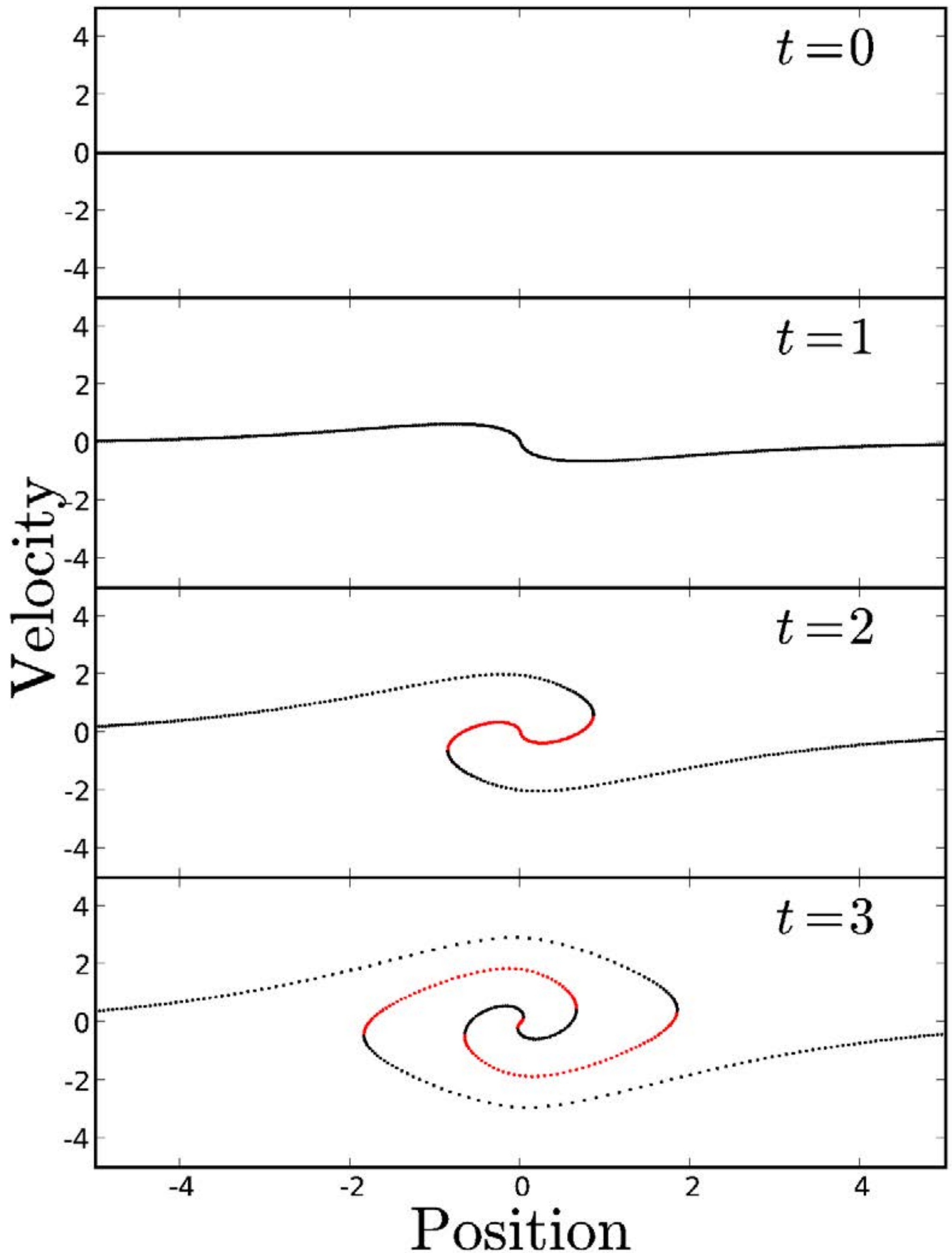
Phase Space Evolution

Dark Matter Phase Space sheet:

3-D structure projection of a folding DM phase space sheet In 6-D phase space

- **Shandarin 2010, 2011**
- **Neyrinck et al. 2011, 2012 Origami**
- **Abel et al. 2011**

Evolving matter distribution in position-velocity space – 1D



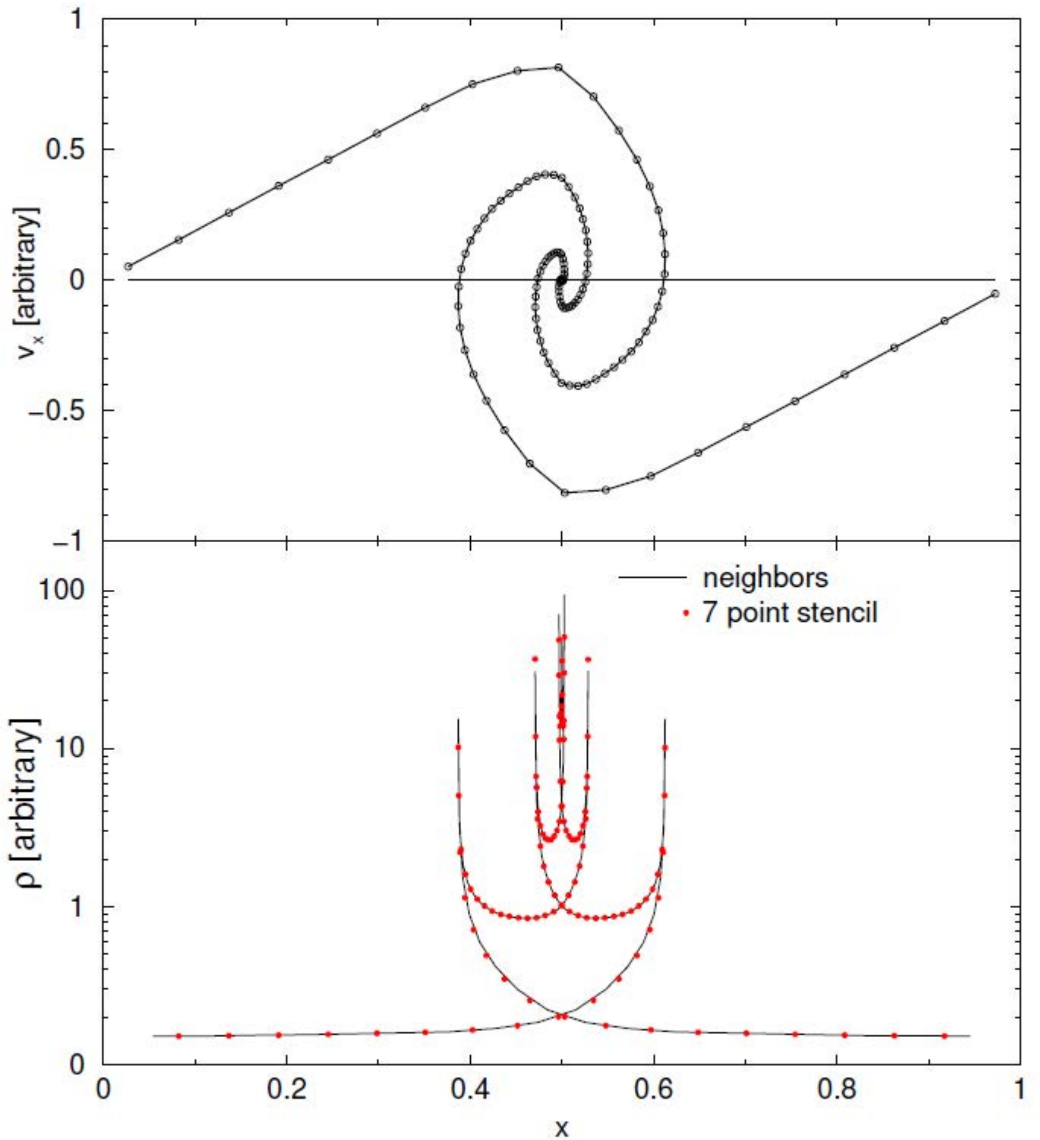
Phase Space Evolution

Phase space:

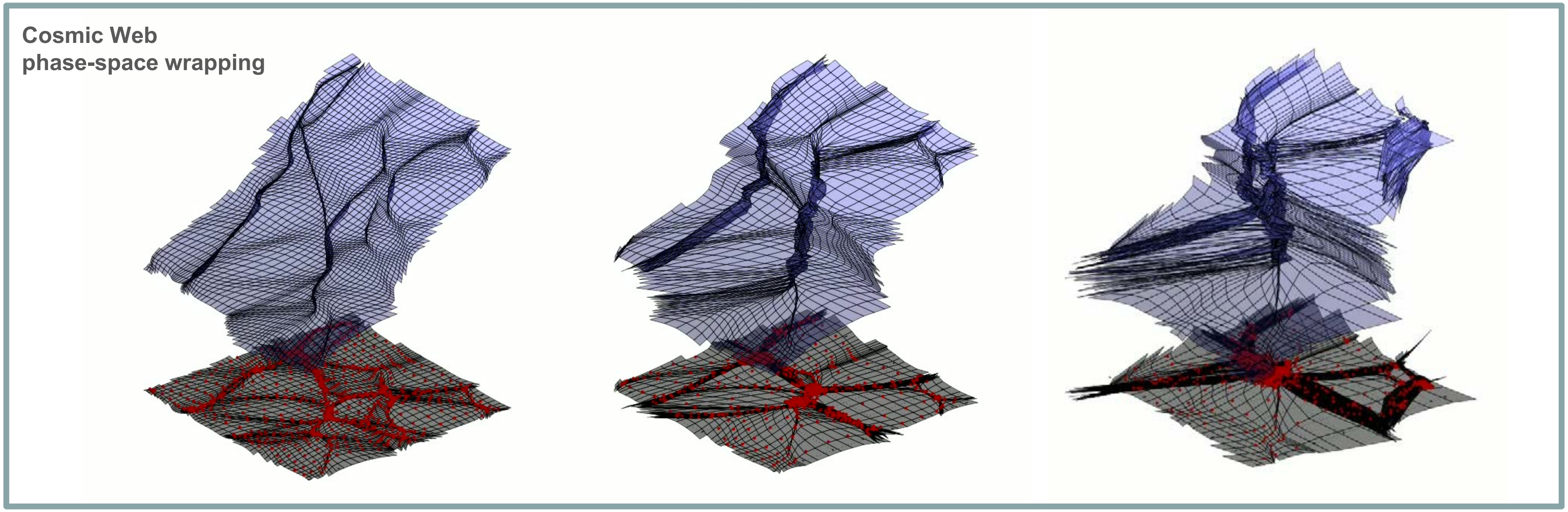
Velocity vs. Position

Density:

$$\rho(\vec{x},t) = \int f\left(\vec{x},\vec{v},t\right) d\vec{v}$$

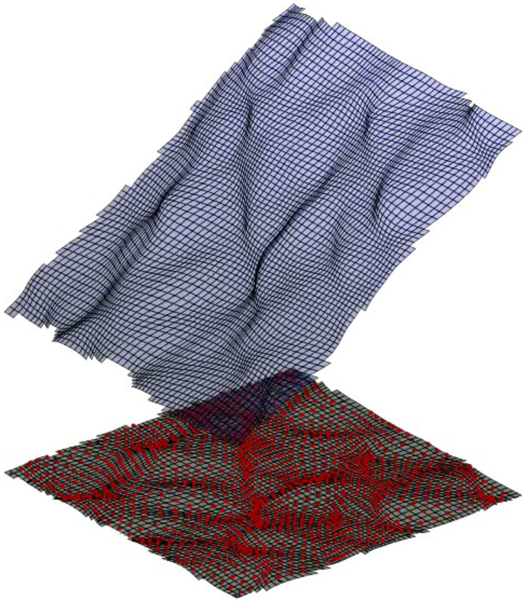


Phase-Space Sheet

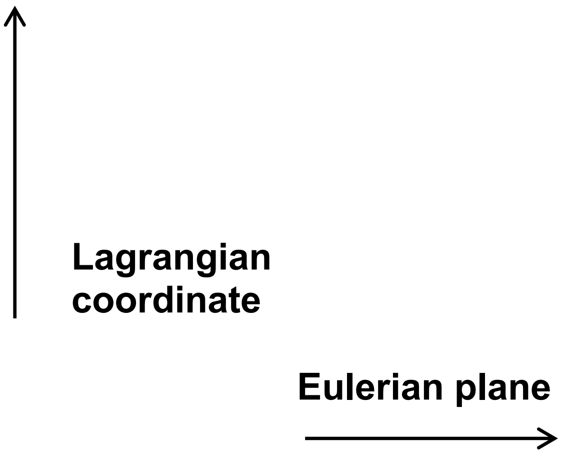


Dynamical Evolution:

folding the
phase-space sheet {q,x}

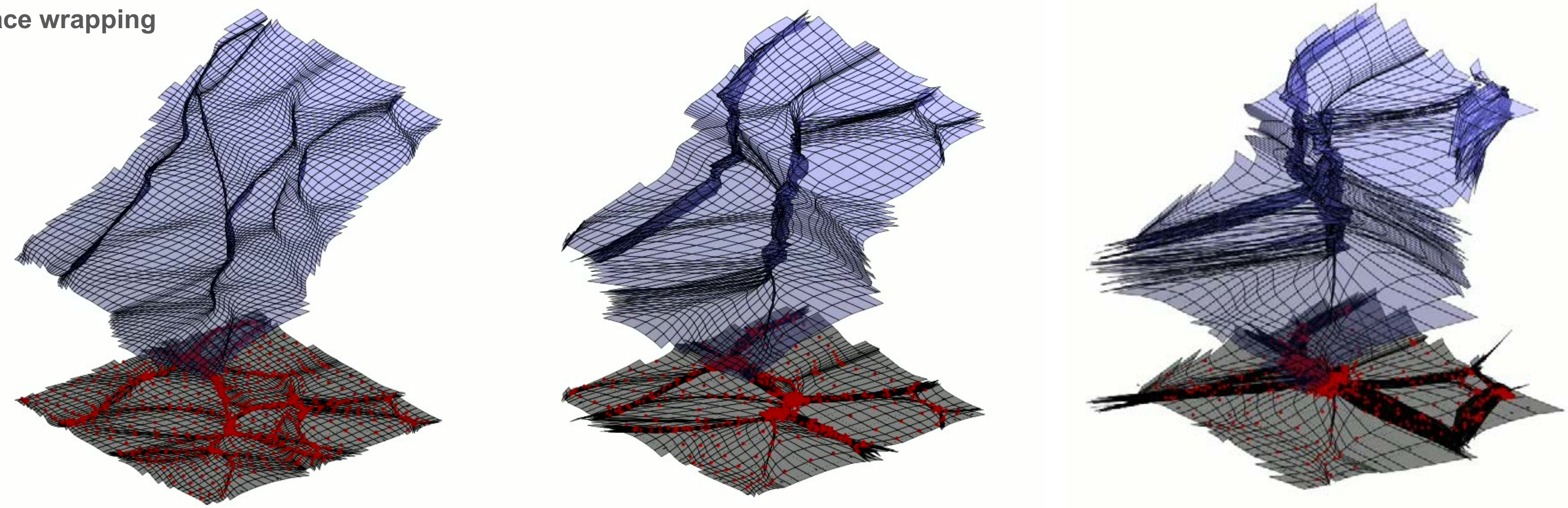


Hidding 2014



Phase-Space Sheet

Cosmic Web
phase-space wrapping



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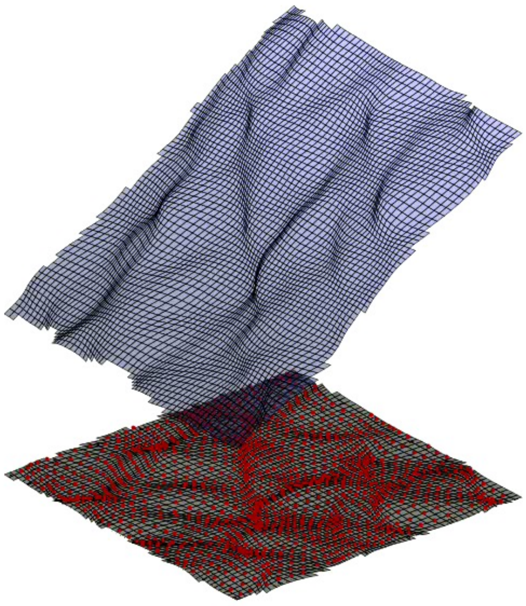


Hidding 2014

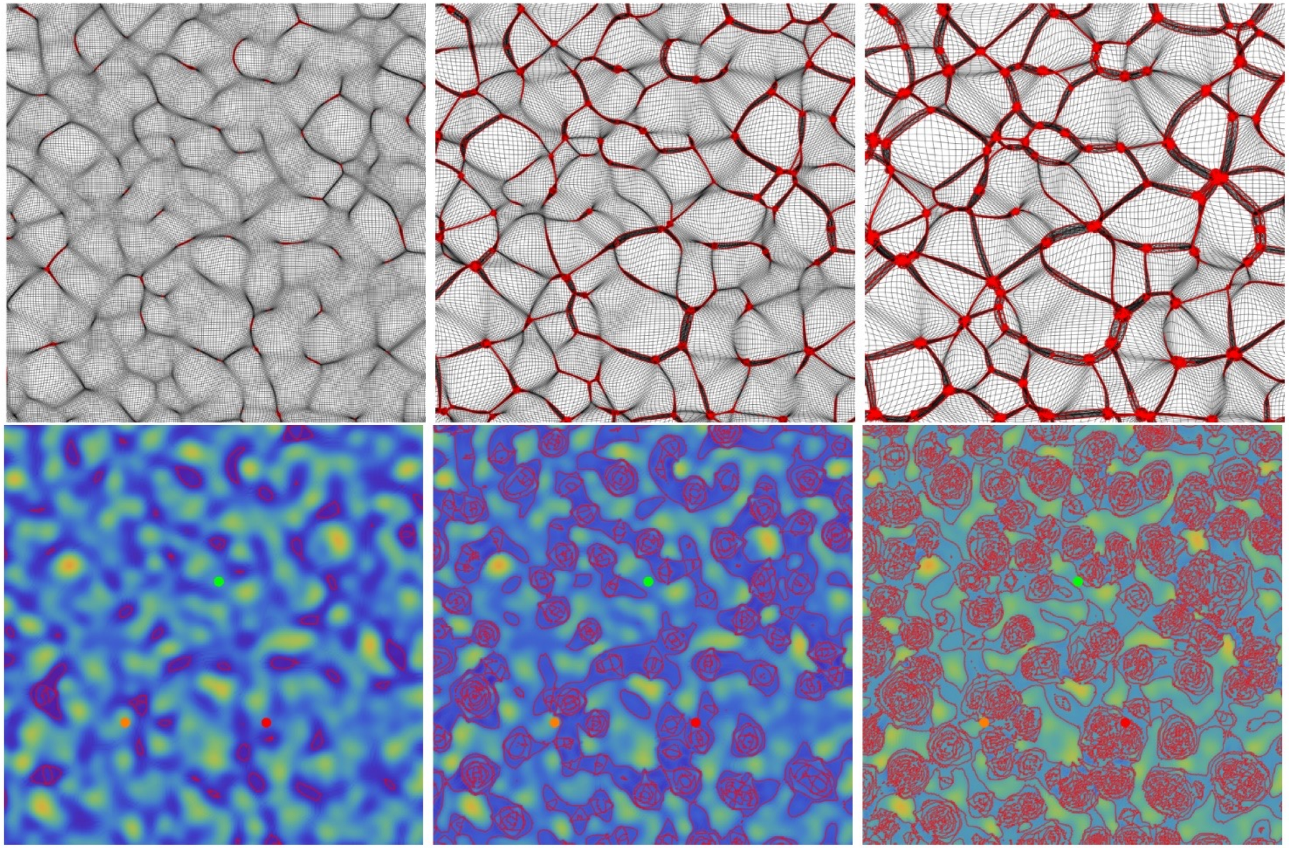
Cosmic Web Multistreaming

Translation towards 2D space:

Evolution Multistream region:

- **Eulerian space**

- **Lagrangian space (mass elements)**



Shandarin 2012
Abel, Hahn & Kaehler 2012
Falck, Neyrinck et al. 2012
Feldbrugge, Wilding, vdW 2022

Cosmic Web FlipFlop field

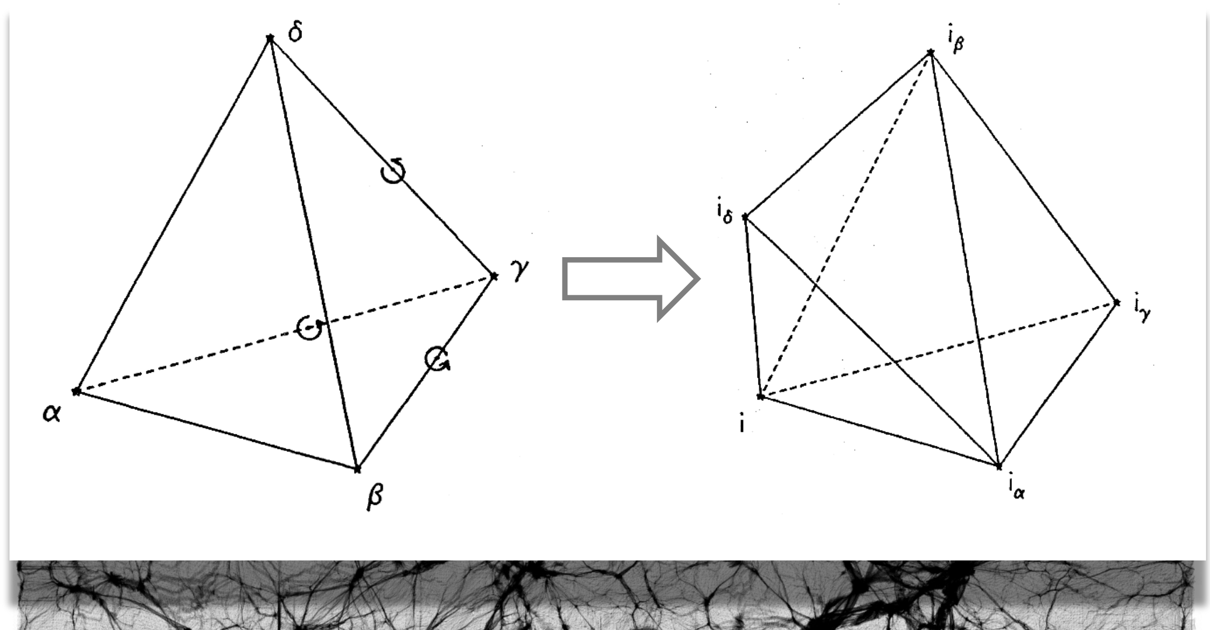
Translation towards 3D space:

Density of dark matter streams:

- **# phase space folds**

=

 # changing orientation tetrahedra



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Feldbrugge et al. 2022b

Cosmic Web FlipFlop field

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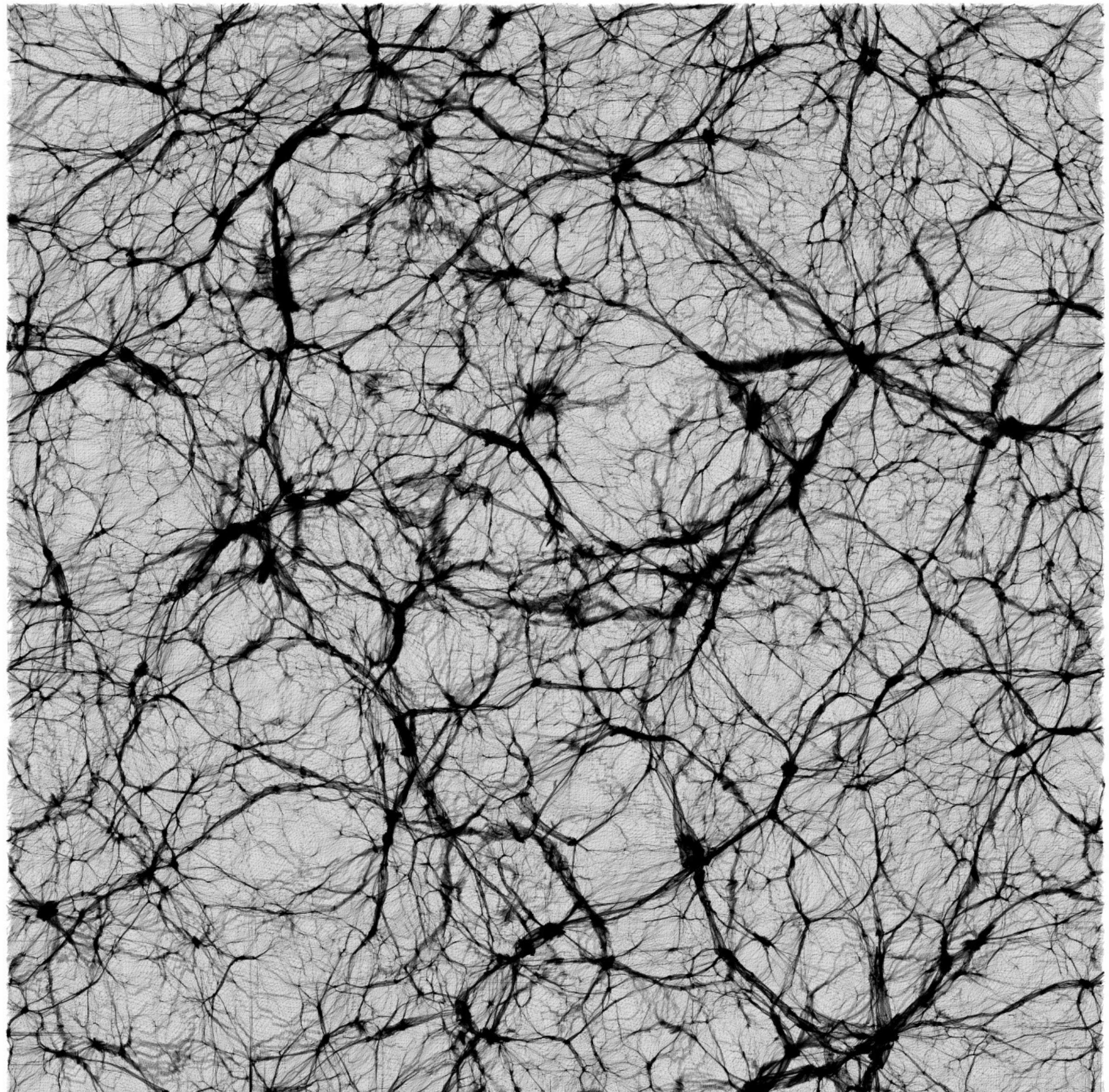

Cosmic Web FlipFlop field

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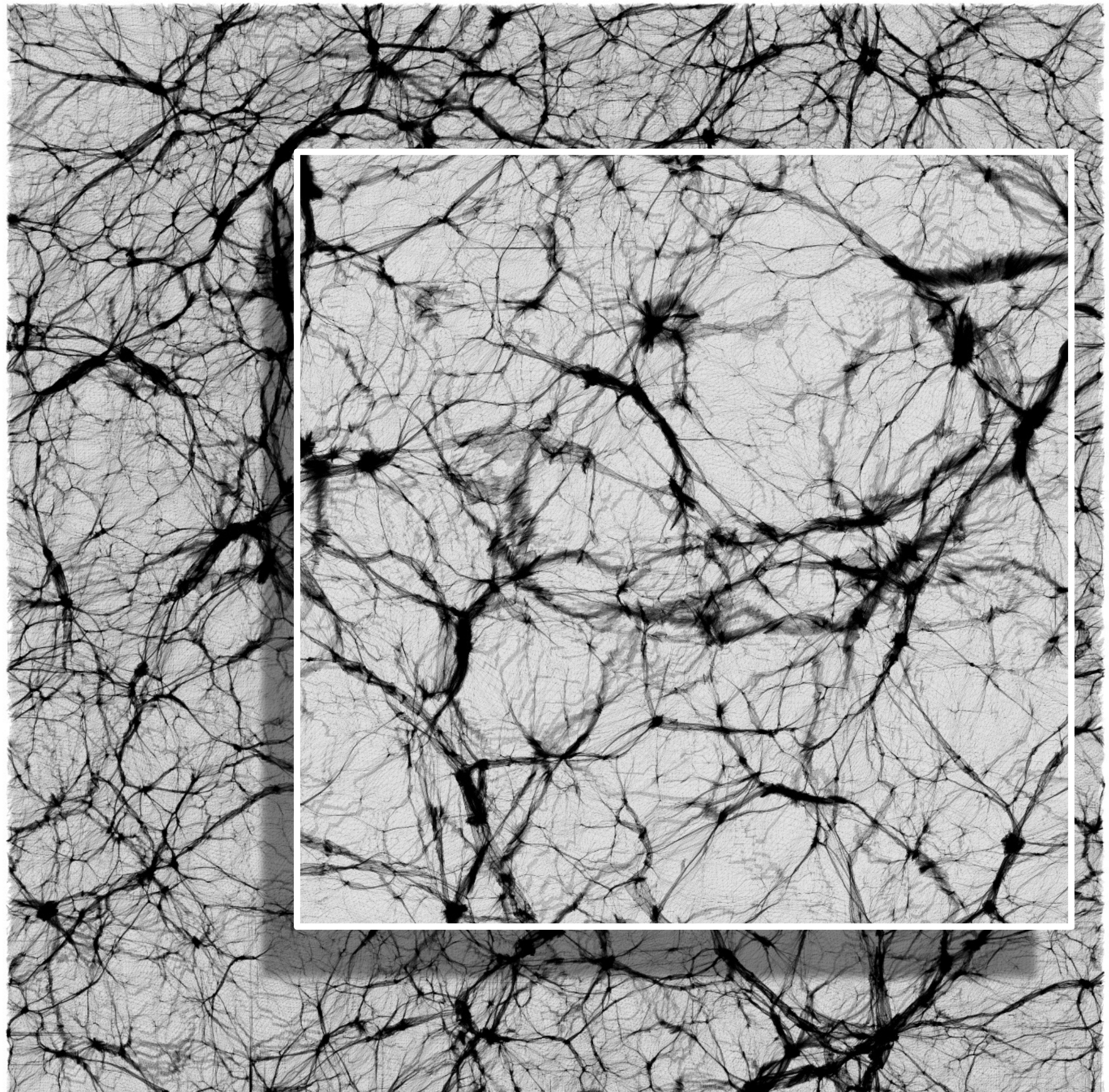
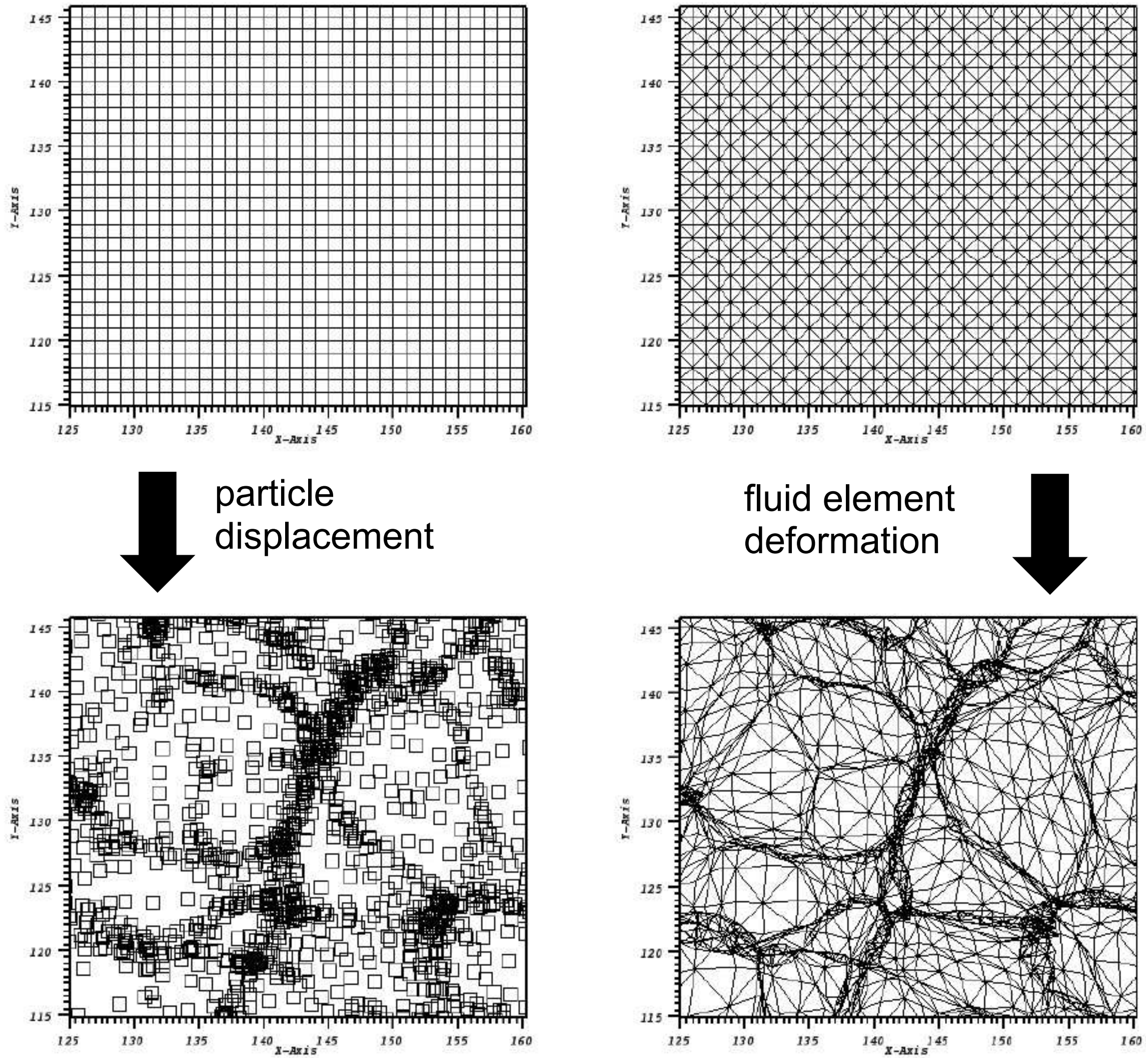


Tessellation Deformation & Phase Space Projection

Translation towards Multi-D space:

- **Look at deformation of initial tessellation**
- **each tessellation cell represents matter cell**
- **evolution deforms cell**
- **once cells start to overlap, manifestation of different phase-space matter streams**

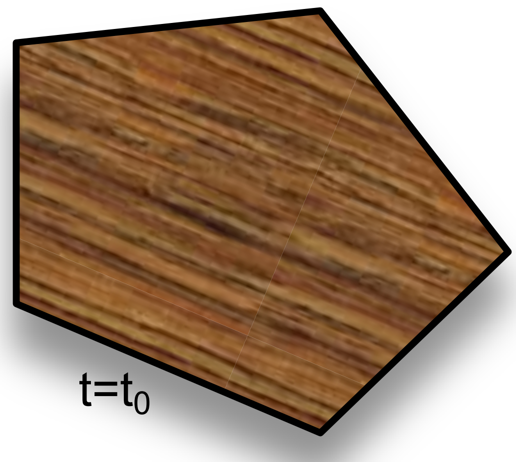


Tessellation Deformation & Phase Space Projection

Translation towards Multi-D space:

- Look at deformation of initial tessellation
- each tessellation cell represents matter cell
- evolution deforms cell

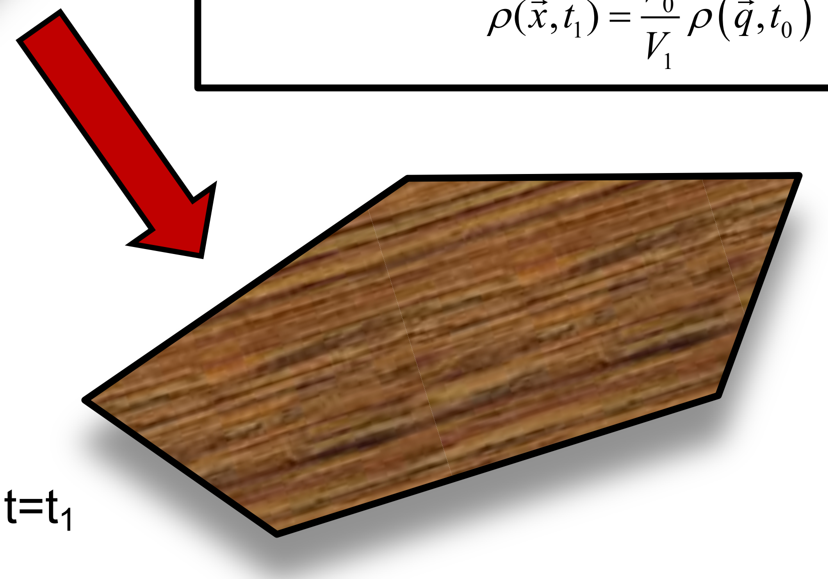
Monostream Density Evolution



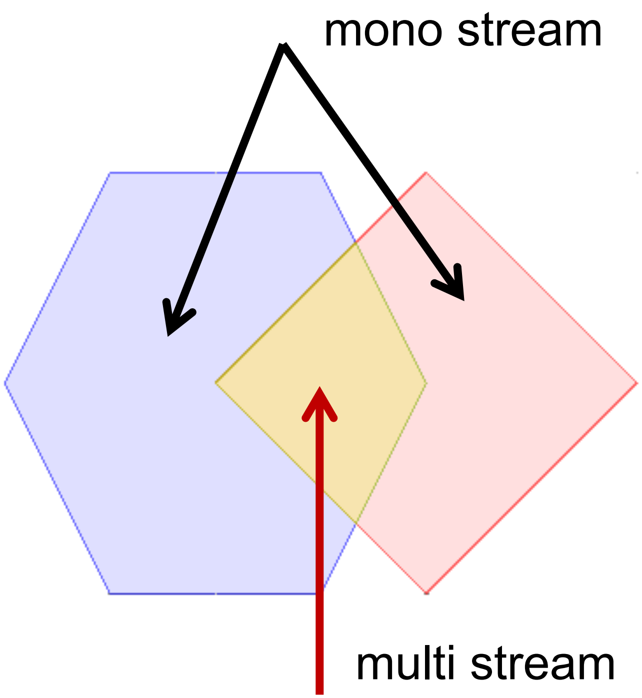
Conservation of mass (continuity eqn.):

$$\rho(\vec{x},t) = \left|J(\vec{x},\vec{q})\right|^{-1} \rho(\vec{q}) = \left|\frac{\partial \vec{x}}{\partial \vec{q}}\right|^{-1} \rho(\vec{q})$$

$$\rho(\vec{x},t_1) = \frac{V_0}{V_1}\,\rho(\vec{q},t_0)$$

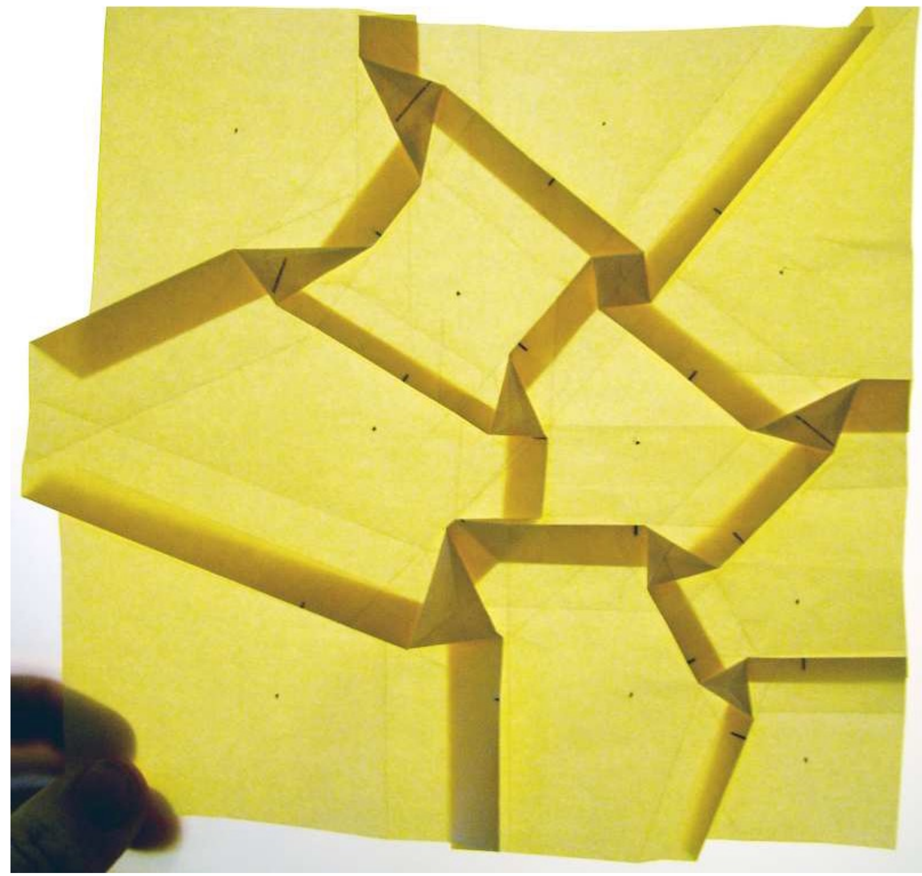


(Cosmic) ORIGAMI



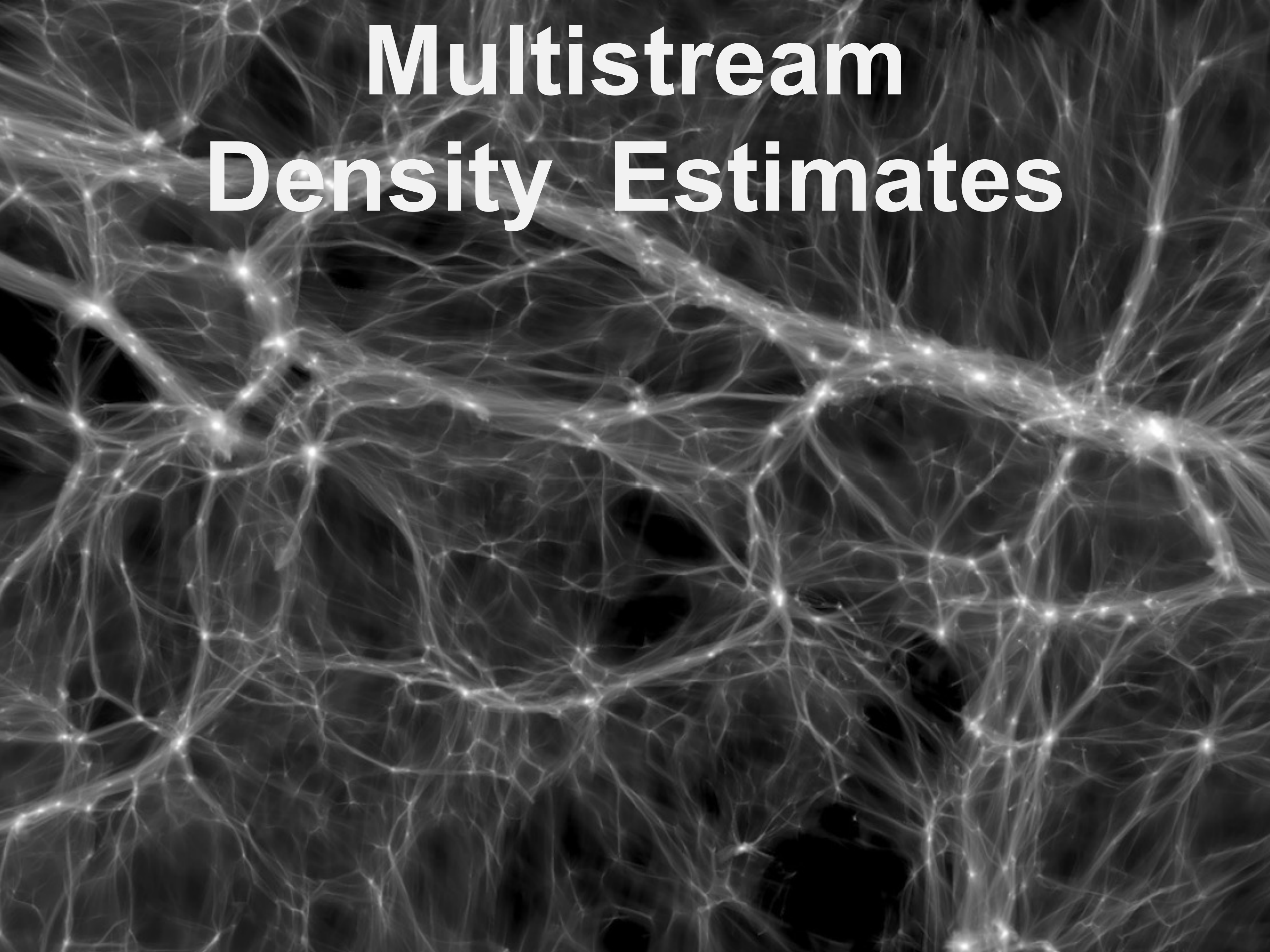
$$\rho_{total}(\vec{x},t_1) = \sum_i \frac{V_{0i}}{V_{1i}} \rho(\vec{q}_i,t_0)$$

Evolution of dynamical system:
Phase-space folding – Cosmic Origami



Mark Neyrinck, Bridget Falck

Multistream Density Estimates



Cosmic Web Stream Density

Translation towards Multi-D space:

Density of dark matter streams:

- # phase space folds

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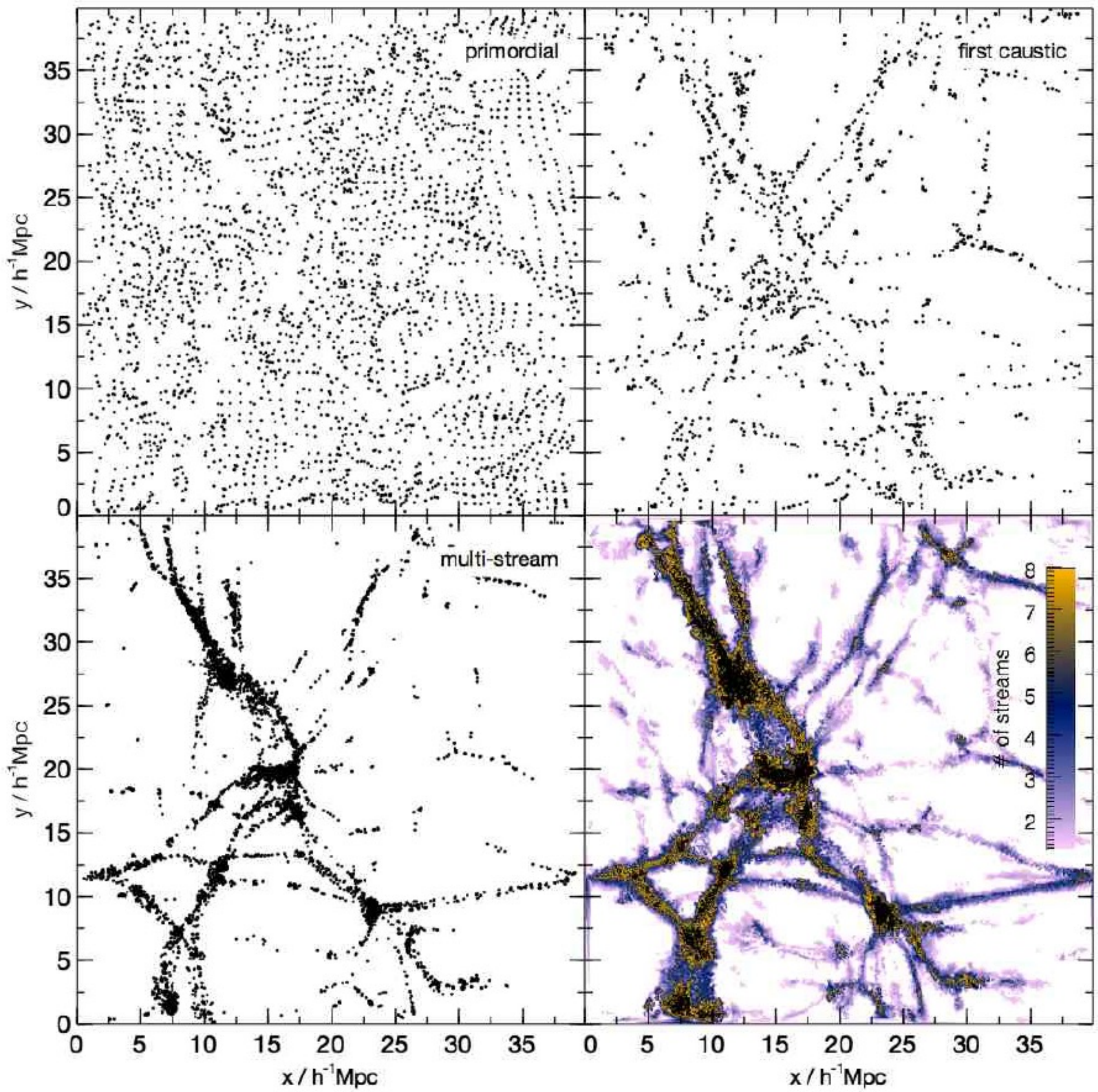
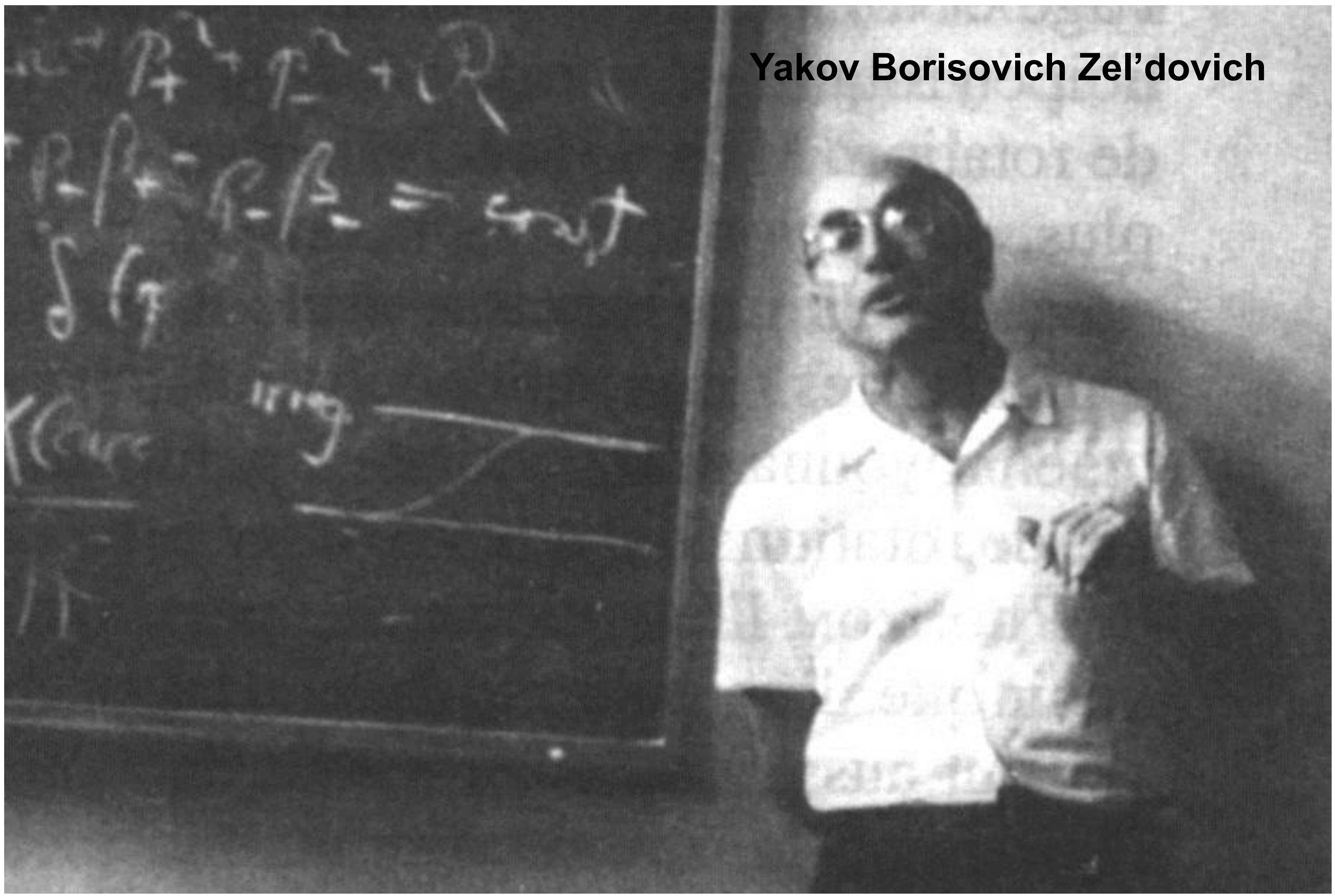
 # locally overlapping tessellation cells

Shandarin 2012
Abel, Hahn & Kaehler 2012
Falck, Neyrinck et al. 2012



1st order Lagrangian Theory:

Zeldovich Formalism



Yakov Borisovich Zel’dovich

Zel'dovich Approximation

1st order Lagrangian perturbation theory

$$\vec{x} = \vec{q} + D(t)\,\vec{u}(\vec{q})$$

$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

$$\Phi(\vec{q}) = \frac{2}{3Da^2H^2\Omega}\phi_{lin}\left(\vec{q}\right)$$

$$\delta_{lin}(\vec{x},t) = D(t)\delta(\vec{x},t_0)$$

linear growth factor D

$$D(t) \approx H(t)\int \frac{dt}{a^2H^2(t)}$$

$$\vec{u}(\vec{q}) = \frac{d\vec{x}}{dD};$$

$$\vec{v}(\vec{q}) = HDf(\Omega)\vec{u}(\vec{q})$$

$$f(\Omega) = \frac{d\log D}{d\log a}$$

linear growth rate f

Zel'dovich Approximation

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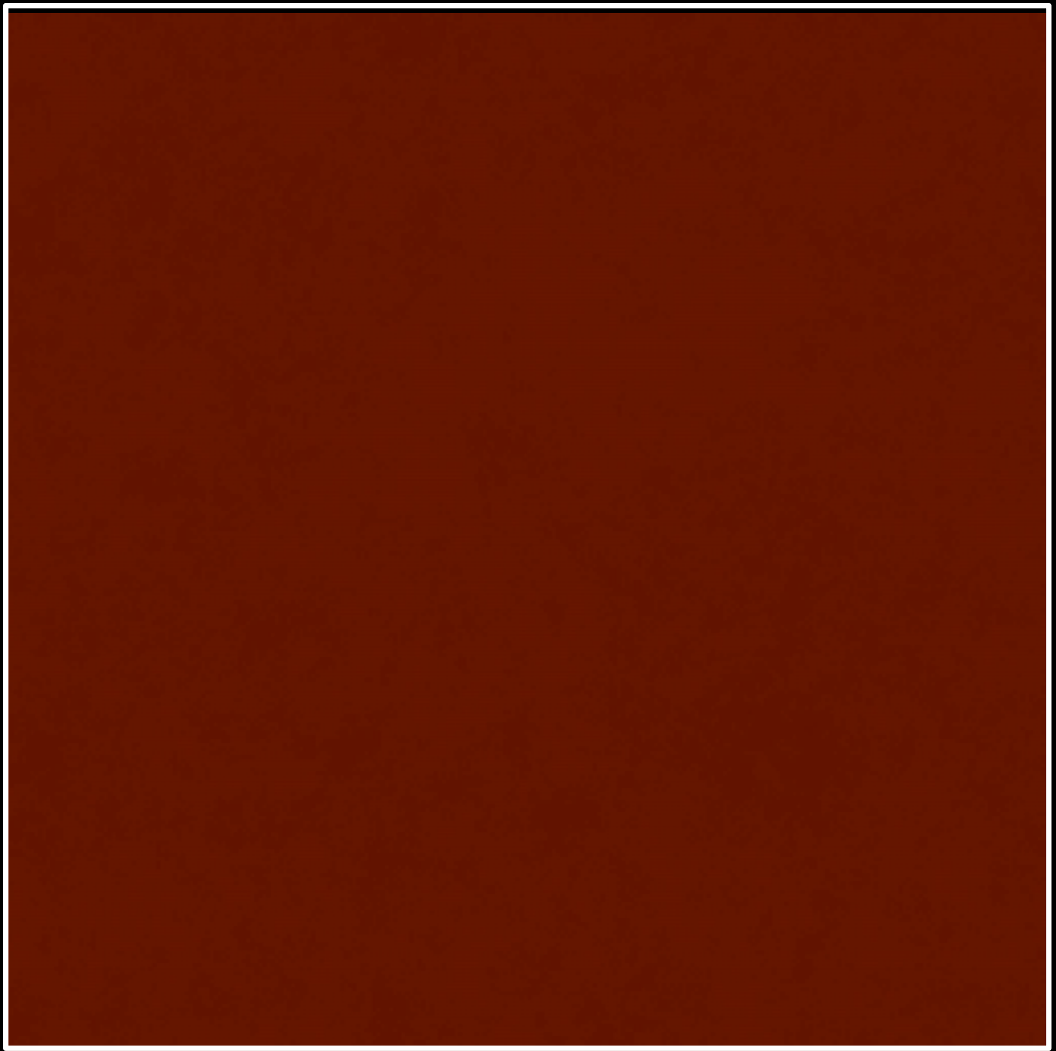

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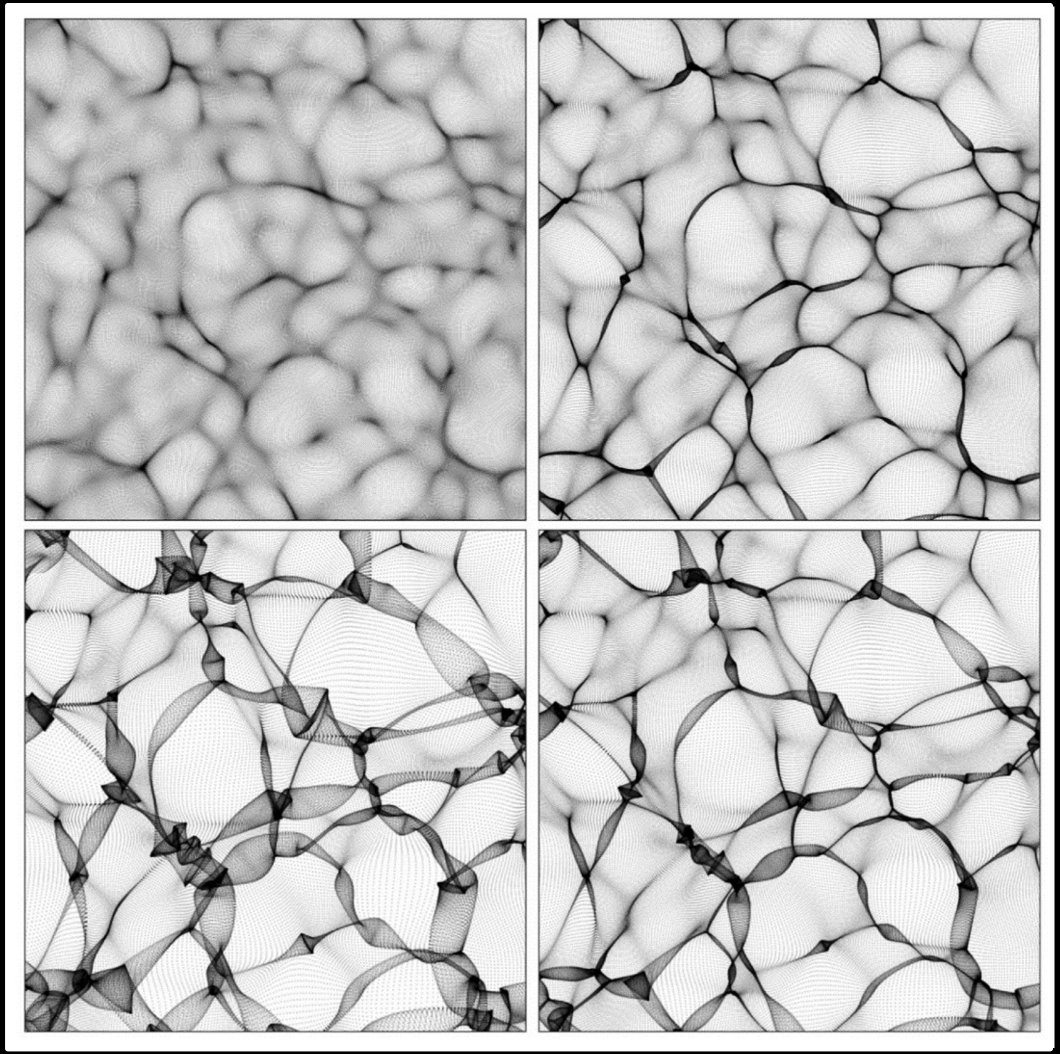


Zel'dovich Approximation: Deformation

$$\vec{x} = \vec{q} + D(t)\vec{u}(\vec{q})$$

$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

$$d_{ij} = -\frac{\partial u_i}{\partial q_j}$$

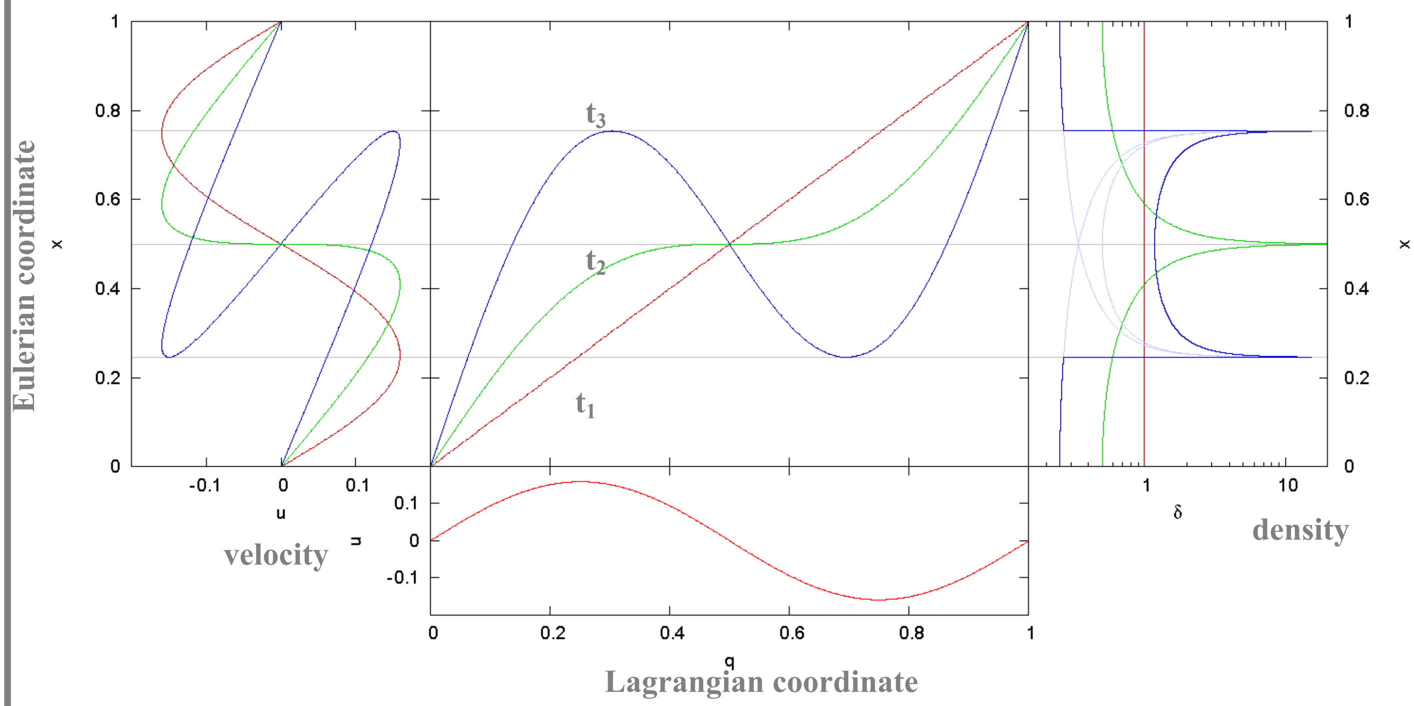
$$\rho(\vec{q},t) = \frac{\rho_u(t)}{\left(1-D(t)\lambda_1(\vec{q})\right)\left(1-D(t)\lambda_2(\vec{q})\right)\left(1-D(t)\lambda_3(\vec{q})\right)}$$

structure of the cosmic web determined by the spatial field of eigenvalues

$$\lambda_1, \lambda_2, \lambda_3$$

Zeldovich Formalism:

Singularities



$$\vec{x}(\vec{q},t) = \vec{q} - D(t)\vec{\nabla}\Phi(\vec{q}) \quad \Rightarrow \quad d_{ij} = \frac{\partial^2 \Phi}{\partial q_i \partial q_j} : \ \lambda_1, \lambda_2, \lambda_3$$

$$\rho(\vec{q},t) = \frac{\rho_u(t)}{\left(1 - D(t)\lambda_1(\vec{q})\right)\left(1 - D(t)\lambda_2(\vec{q})\right)\left(1 - D(t)\lambda_3(\vec{q})\right)}$$

Zel'dovich Morphology

$$\rho(\vec{q},t) = \frac{\rho_u(t)}{\left(1-D(t)\lambda_1(\vec{q})\right)\left(1-D(t)\lambda_2(\vec{q})\right)\left(1-D(t)\lambda_3(\vec{q})\right)}$$

$$\lambda_1, \lambda_2, \lambda_3 \qquad\qquad \lambda_1 > \lambda_2 > \lambda_3$$

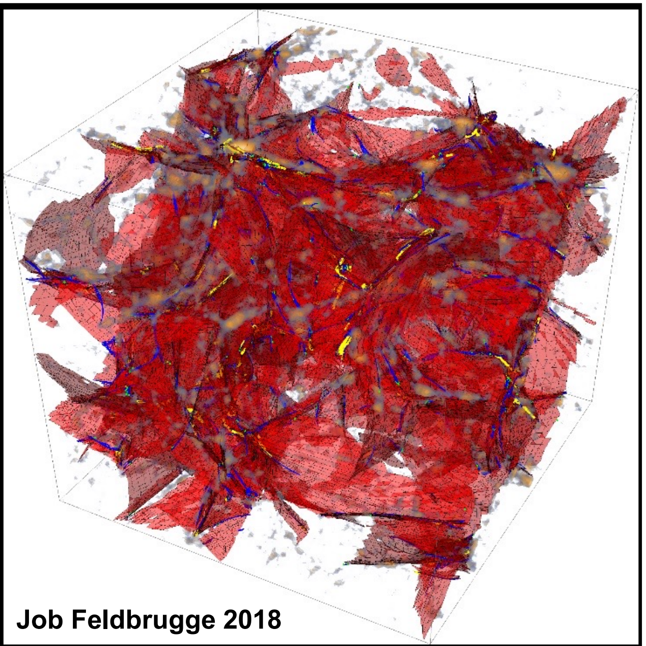
Structure of the cosmic web determined by the spatial field of eigenvalues:

<u>Sequence of formation stages:</u>

λ1 - collapse along first axis:
formation of walls/sheets/pancakes

λ2 - collapse along 2 axes:
formation of elongated filaments

λ3 - possibly – if λ3>0 – collapse along all three axes, into a fully collapsed clump/node



Job Feldbrugge 2018

Zel'dovich Cosmic Web

It is no exaggeration to state that Zeldovich (1970) predicted the existence of the Cosmic Web !

Sequence of formation stages:

λ1 - collapse along first axis:
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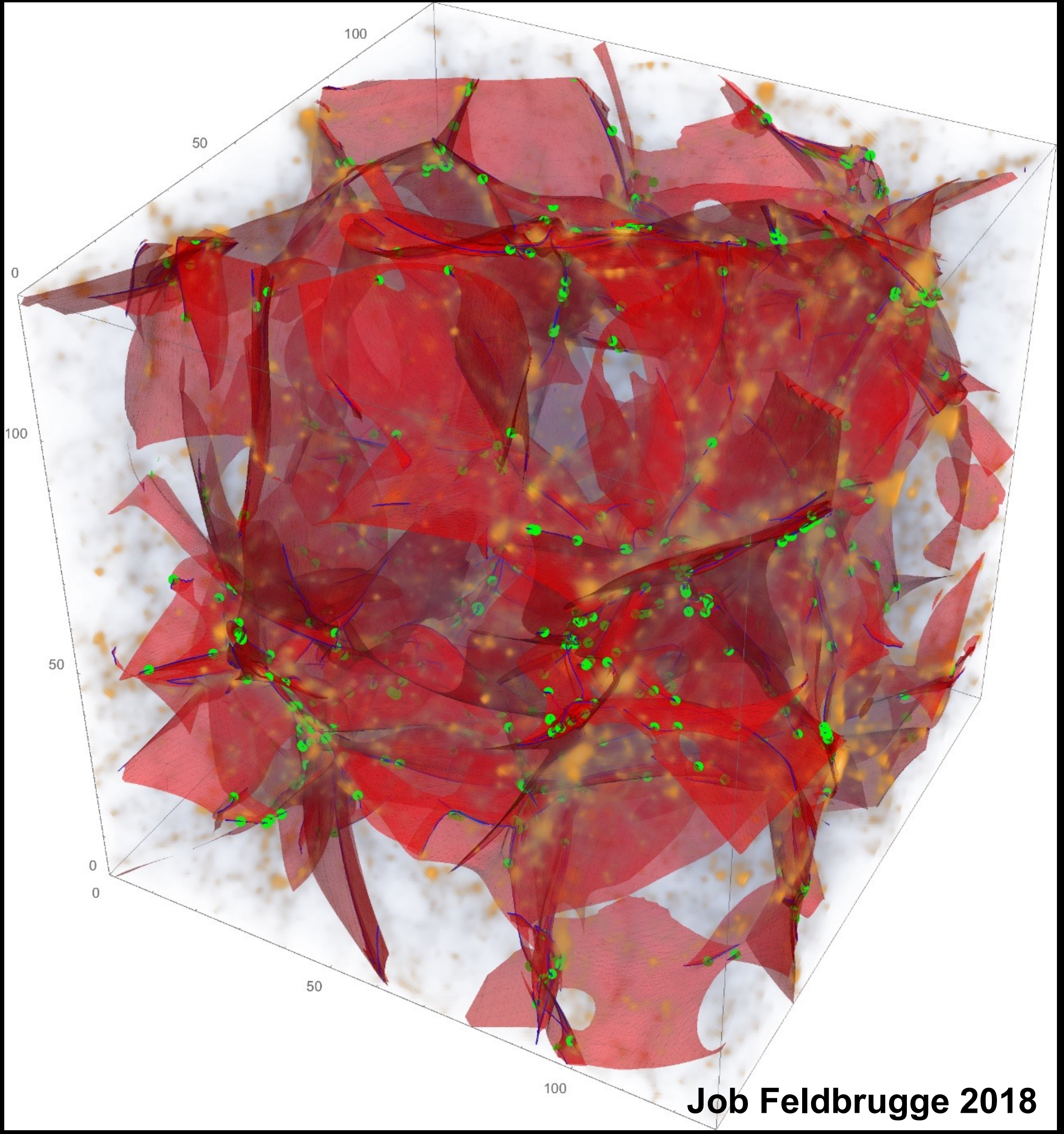


Job Feldbrugge 2018

Yakov Borisovich Zel’dovich

Zel'dovich Dynamics

1st order Lagrangian perturbation theory

$$\vec{x} = \vec{q} + D(t)\,\vec{u}(\vec{q})$$

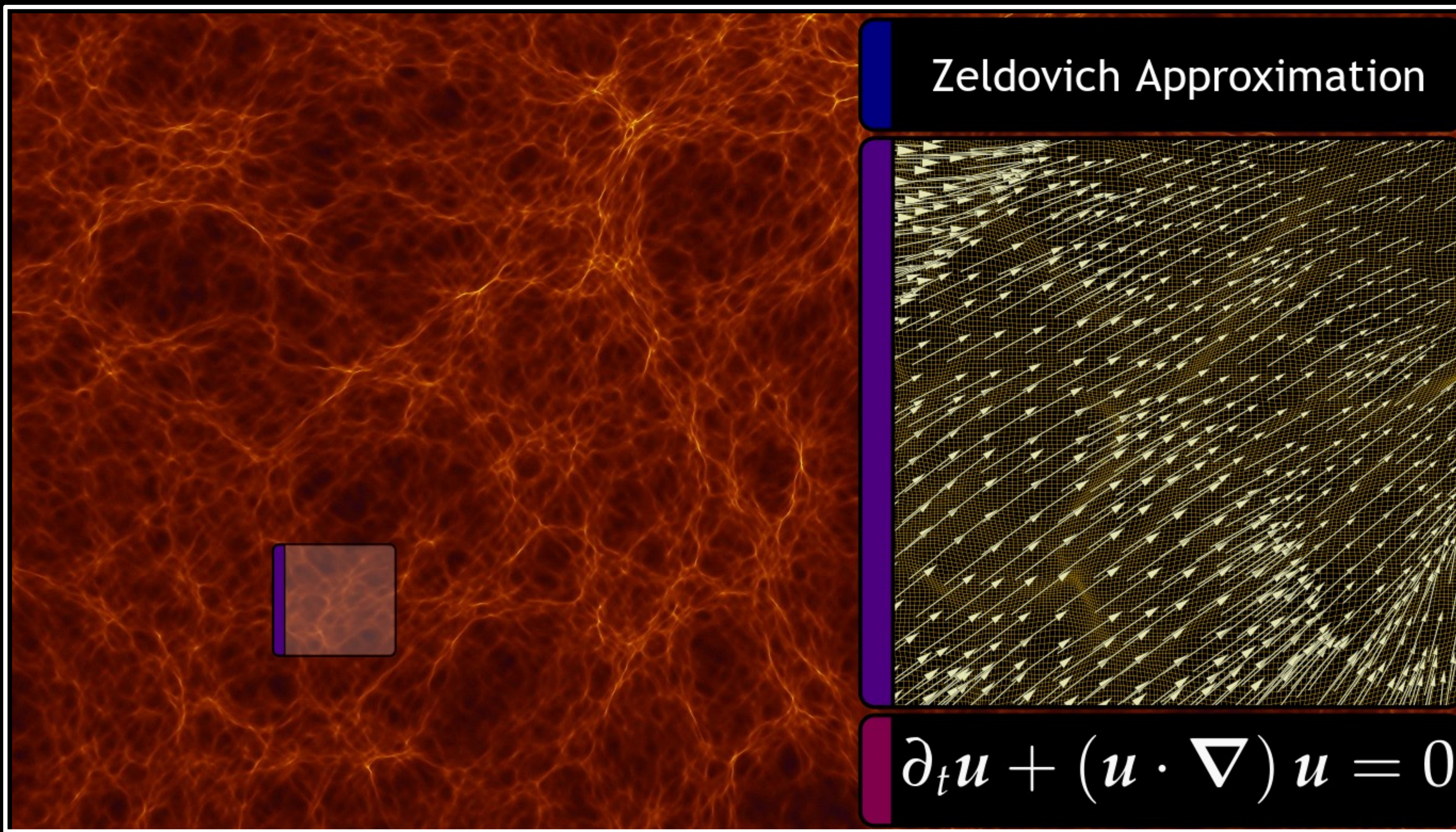
$$\vec{u}(\vec{q}) = -\vec{\nabla}\Phi(\vec{q})$$

Euler equation:

force-free flow

$$\frac{\partial \vec{u}}{\partial D} + \left(\vec{u}\cdot\nabla\right)\vec{u} = 0$$

Zel'dovich Dynamics



Zeldovich Dynamics

By rewriting the Euler equation (in comoving coordinates), we may easily understand dynamical nature of the Zeldovich approximation:

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} + \frac{1}{a}\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\frac{1}{a}\vec{\nabla}\phi$$

Define velocity u,
wrt linear growth factor D(t):

$$\vec{u} = \frac{d\vec{x}}{dD} = \frac{\vec{v}}{a\dot{D}}$$

Zeldovich Dynamics

Following some algebraic manipulations, one arrives at the equivalent Euler equation for the normalized velocity u:

$$\frac{\partial \vec{u}}{\partial D} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} = -\vec{\nabla}\left(\frac{3\Omega}{2f^2 D}\phi_v + \frac{\phi}{a^2 \dot{D}^2}\right) = -\vec{\nabla} V$$

With velocity potential ϕ_v:

$$\vec{u} = -\vec{\nabla}\phi_v$$

and effective potential V:

$$V = \frac{3\Omega}{2f^2 D}\left(\phi_v + \theta\right)$$

and scaled gravitational potential θ:

$$\theta = \frac{2\phi}{3\Omega a^2 D H^2}$$

Effective & Scaled Potentials

For the Zeldovich approximation we may easily see that the effective potential V=0:

$$V = \frac{3\Omega}{2f^2 D}\left(\phi_v + \theta\right) = 0$$

For the Zeldovich approximation:

$$\vec{x} = \vec{q} - D(t)\vec{\nabla}\Psi(\vec{q})$$

with:

$$\Psi(\vec{q}) = \frac{2}{3Da^2H^2\Omega}\phi\left(\vec{x},t\right)$$

so that the scaled gravitational potential θ:

$$\theta = \frac{2\phi}{3\Omega a^2 DH^2} = \Psi\left(\vec{q}\right)$$

The velocity potential ϕ_v we may infer from the velocity corresponding to the Zeldovich approximation:

$$\vec{v} = a\dot{\vec{x}} = -aDH\, f\left(\Omega\right)\vec{\nabla}\Psi\left(\vec{q}\right)$$

$$\vec{u} = \vec{\nabla}\phi_v = \frac{\vec{v}}{a\dot{D}} = -\frac{aDH}{a\dot{D}}\, f\left(\Omega\right)\vec{\nabla}\Psi\left(\vec{q}\right) = -\vec{\nabla}\Psi\left(\vec{q}\right)$$

from which we see that

$$\phi_v = -\Psi\left(\vec{q}\right)$$

Hence, for the Zeldovich approximation: $\phi_v + \theta = 0 \quad \Rightarrow \quad V = 0$

Zel’dovich ++:

Adhesion Formalism
&
Hierarchical Cosmic Web Dynamics

Zeldovich-Adhesion

We saw that dynamically, the Zeldovich approximation corresponds to a force-free propagation, as evidenced by the Euler equation for the normalized velocity u:

$$\frac{\partial \vec{u}}{\partial D} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} = -\vec{\nabla} V = 0$$

The force-free nature of the Zeldovich approximation leads to the ballistic motion, which once a mass element enters a multi-stream nonlinear region ignores the dominant self-gravitational terms, ie. the evolving gravitational potential of high-density structures (such as walls, filaments and clumps).

The adhesion approximation augments this with a (really) artificial term – a non-gravitational term – in terms of a viscosity term (as we know from the Navier-Stokes equation):

$$\frac{\partial \vec{u}}{\partial D} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} = \nu \nabla^2 \vec{u}$$

Zeldovich-Adhesion

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$$\frac{\partial \vec{u}}{\partial D} + \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} = \nu \, \nabla^2 \vec{u}$$

This equation, the Navier-Stokes equation for a pressureless medium, goes by the name of

Burger's Equation

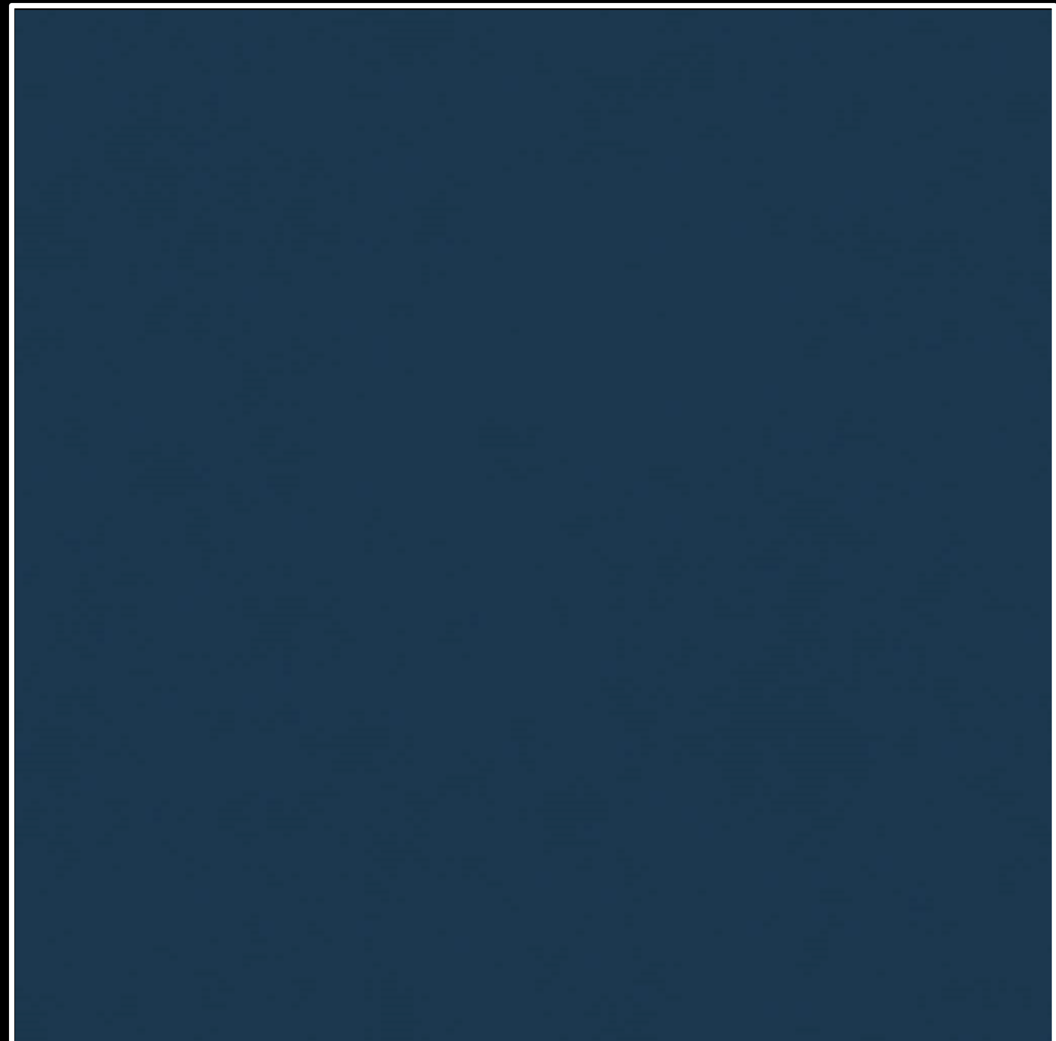
after the famous hydrodynamicist. It is one of the few equations that can be fully solved analytically.

The viscosity term here is fully artificial, tries to emulate "selfgravity", and has nothing to do with the physical viscosity we know from hydrodynamics. Basically, it functions as a friction term.

In its cosmological context, you only want to invoke it close to the emerging multistream regions, so that you take the asymptotic "inviscid" limit, $\nu \rightarrow 0$

Adhesion Approximation

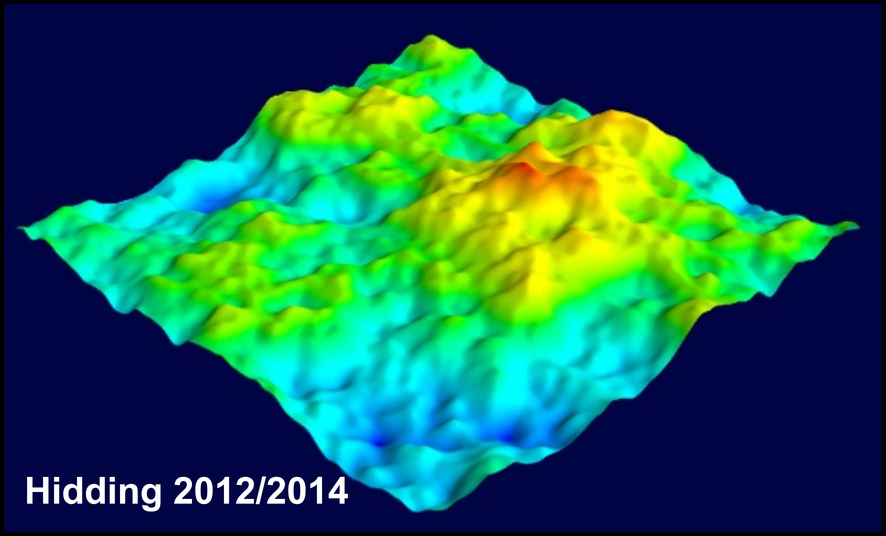
Gurbatov, Saichev & Shandarin 1987

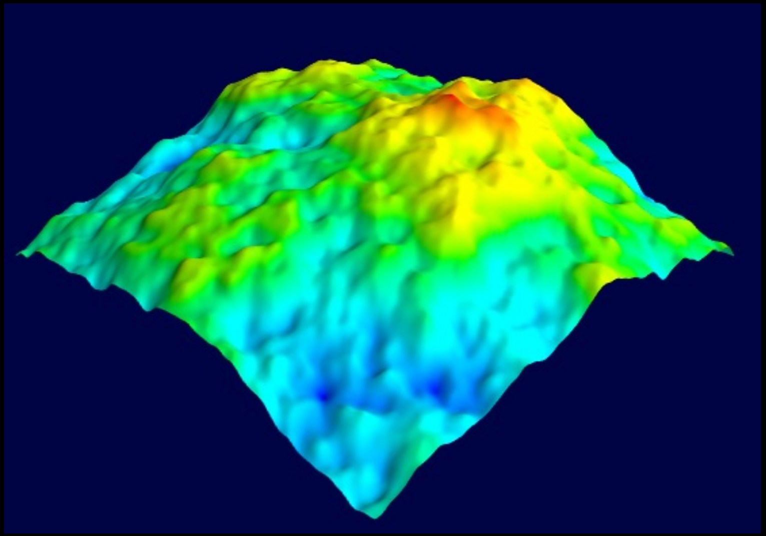


Hidding 2012

Burger's Equation: Hopf Solution

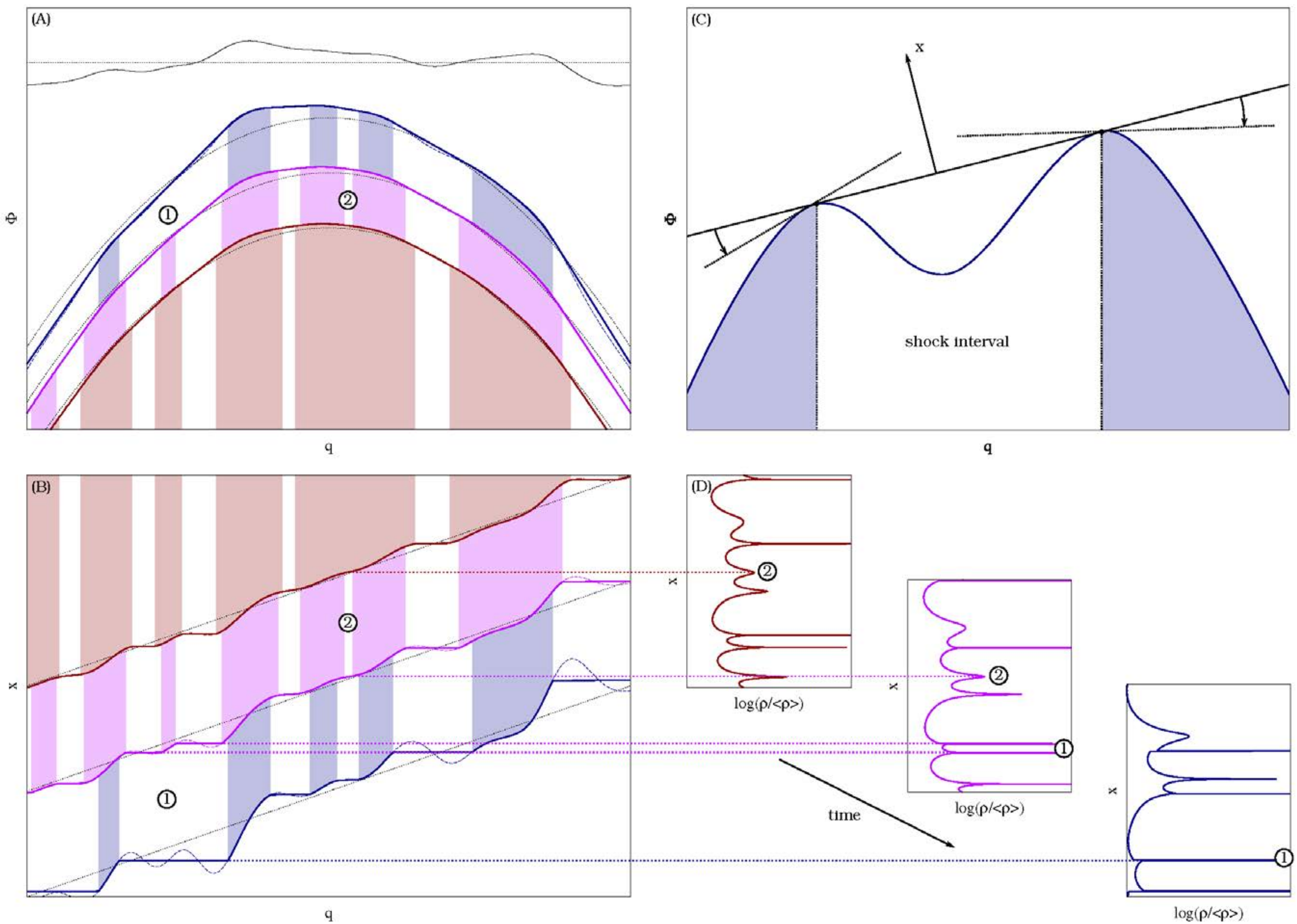
$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} = \nu \nabla^2 \vec{u}$$

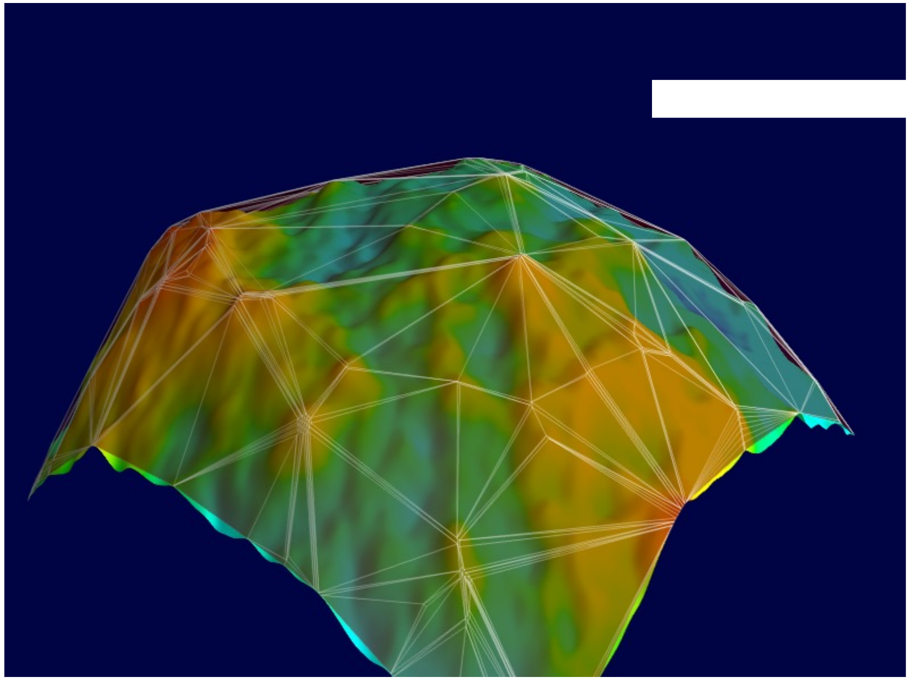


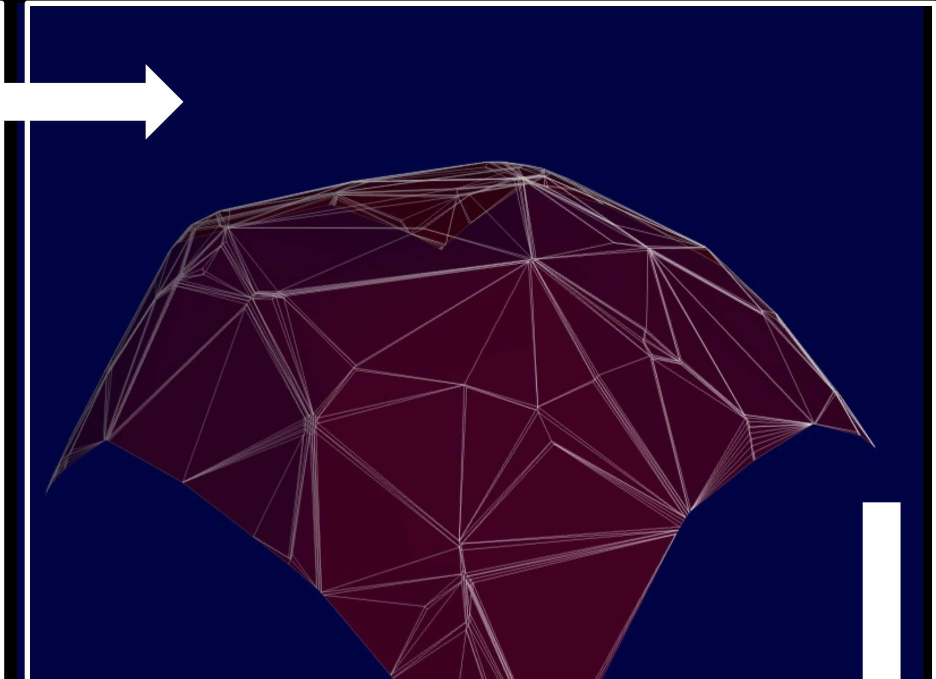


$$\Phi(\vec{x},t) + \frac{x^2}{2} = \max_q \left[\left(t\Phi_0(q) - \frac{q^2}{2} \right) + \vec{x} \cdot \vec{q} \right]$$

Burger's Equation: Hopf Solution



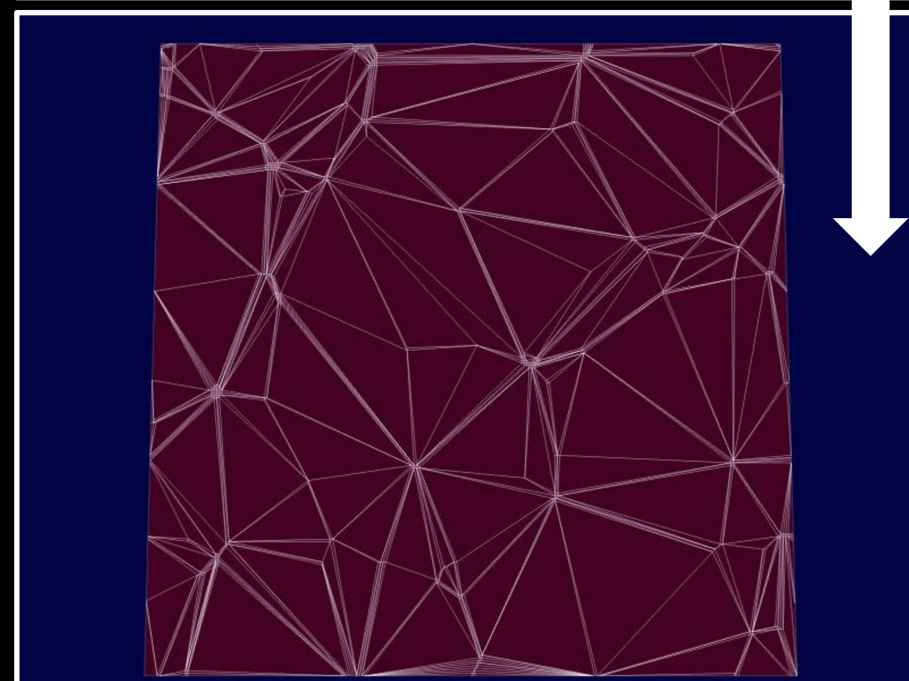




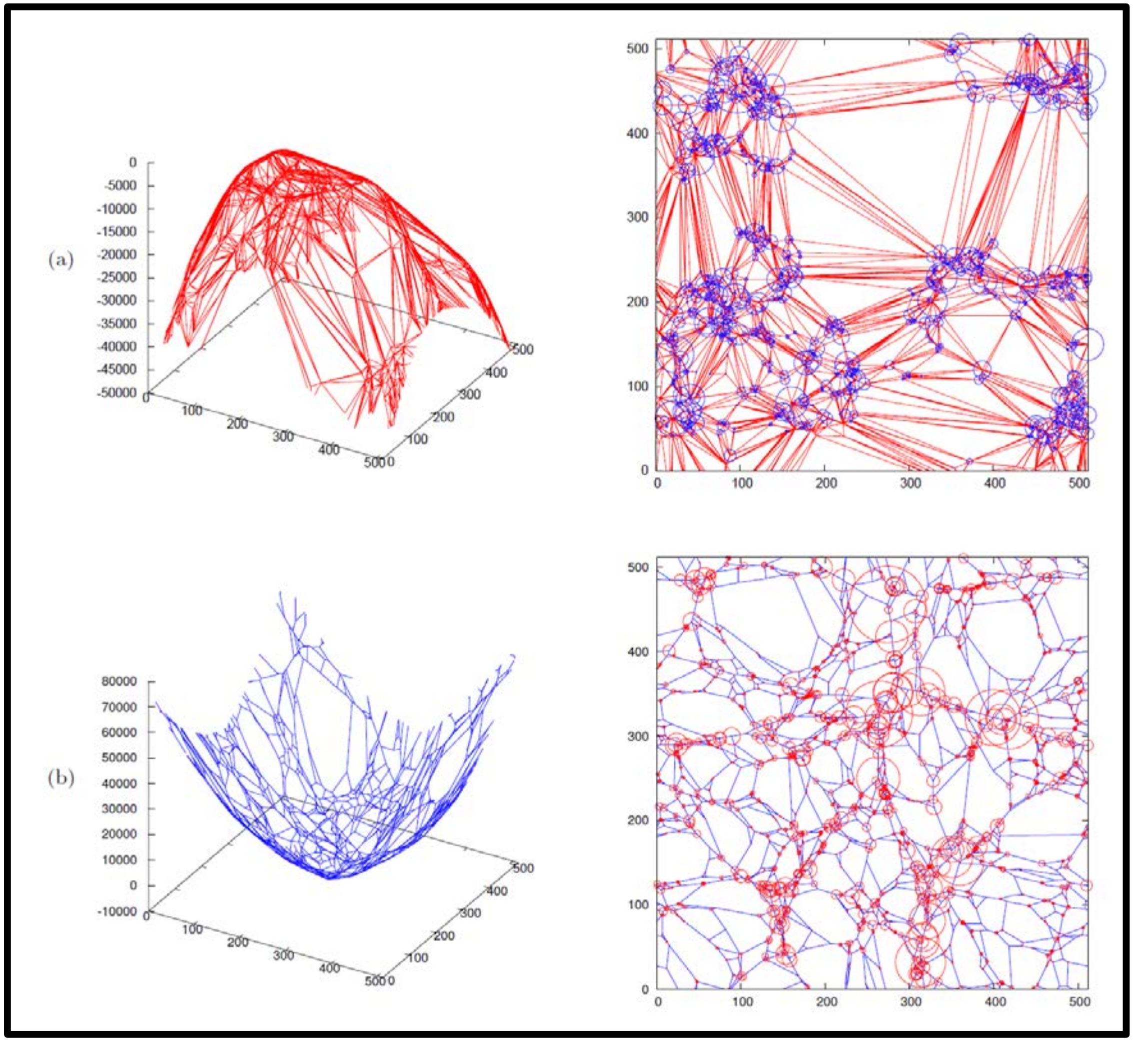
Hidding 2012/2014

Convex Hull
quadratically lifted potential field

Delaunay tessellation
generated by maxima potential field



Convex Hull
Delaunay-Voronoi: Legendre transform



Delaunay

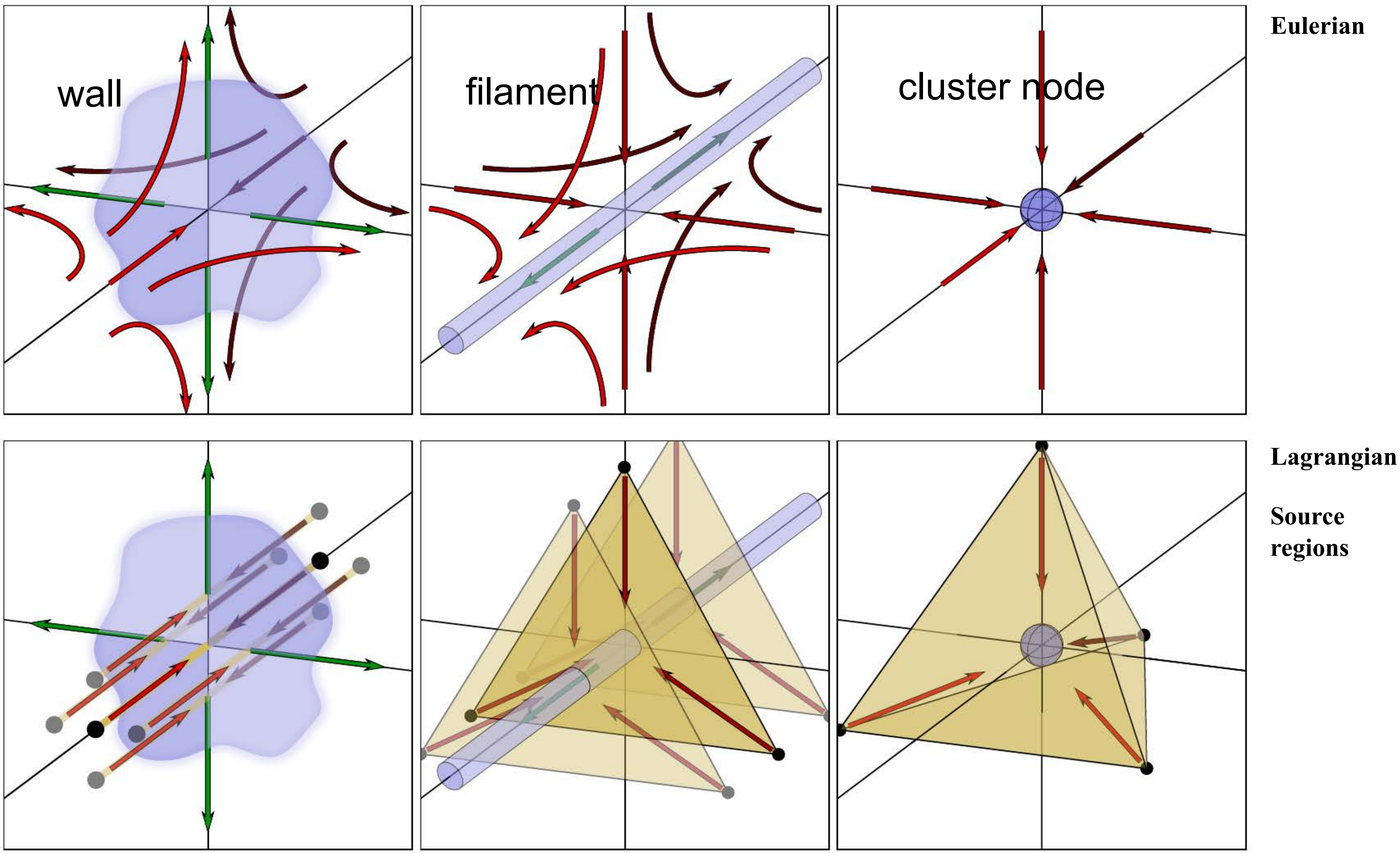
(weighted)

Voronoi

(weighted)

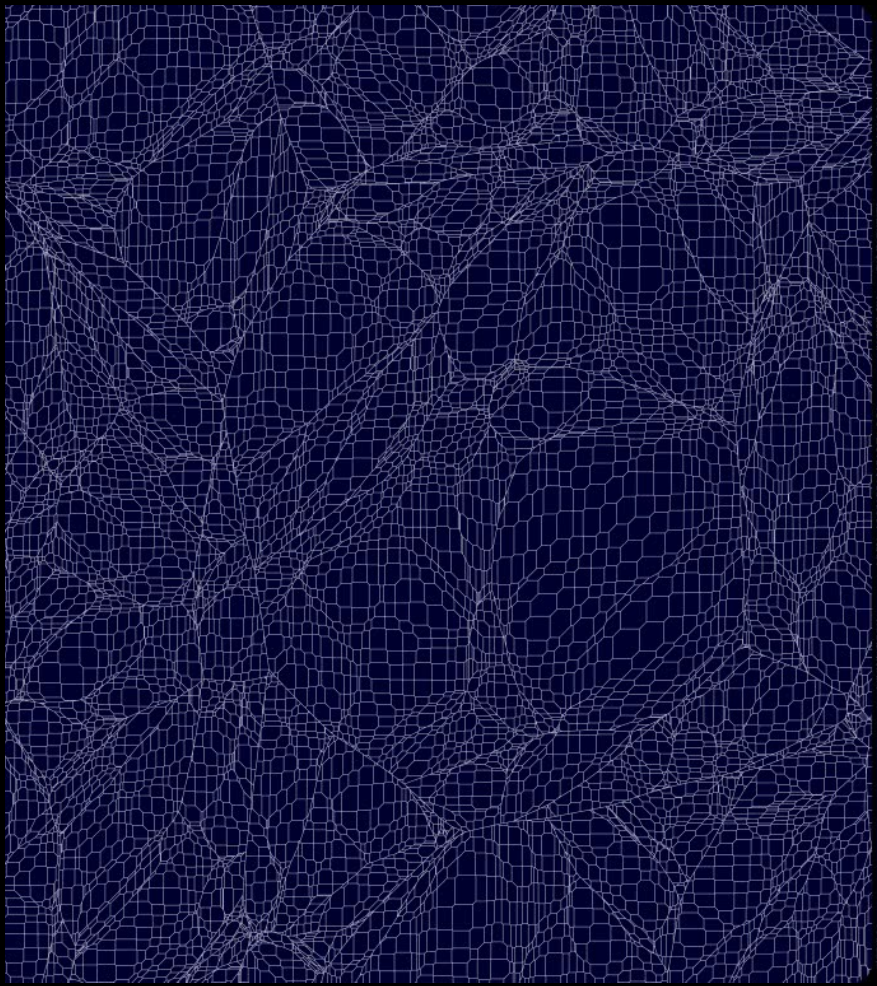
$$V_q = \left\{ \vec{x} \in E \middle| (\vec{x} - \vec{q})^2 + w_q \leq (\vec{x} - \vec{p})^2 + w_p ,\ \forall\, \vec{p} \in L \right\}$$

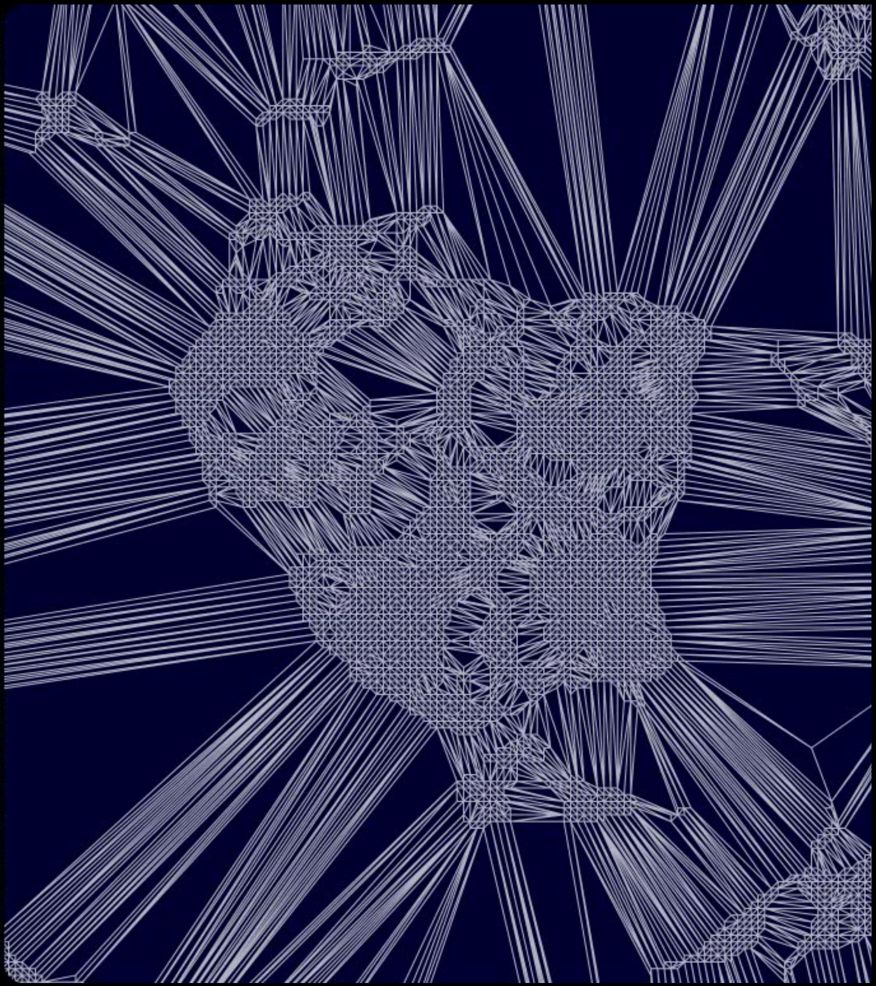
Euler vs. Lagrangian
weblike geometry



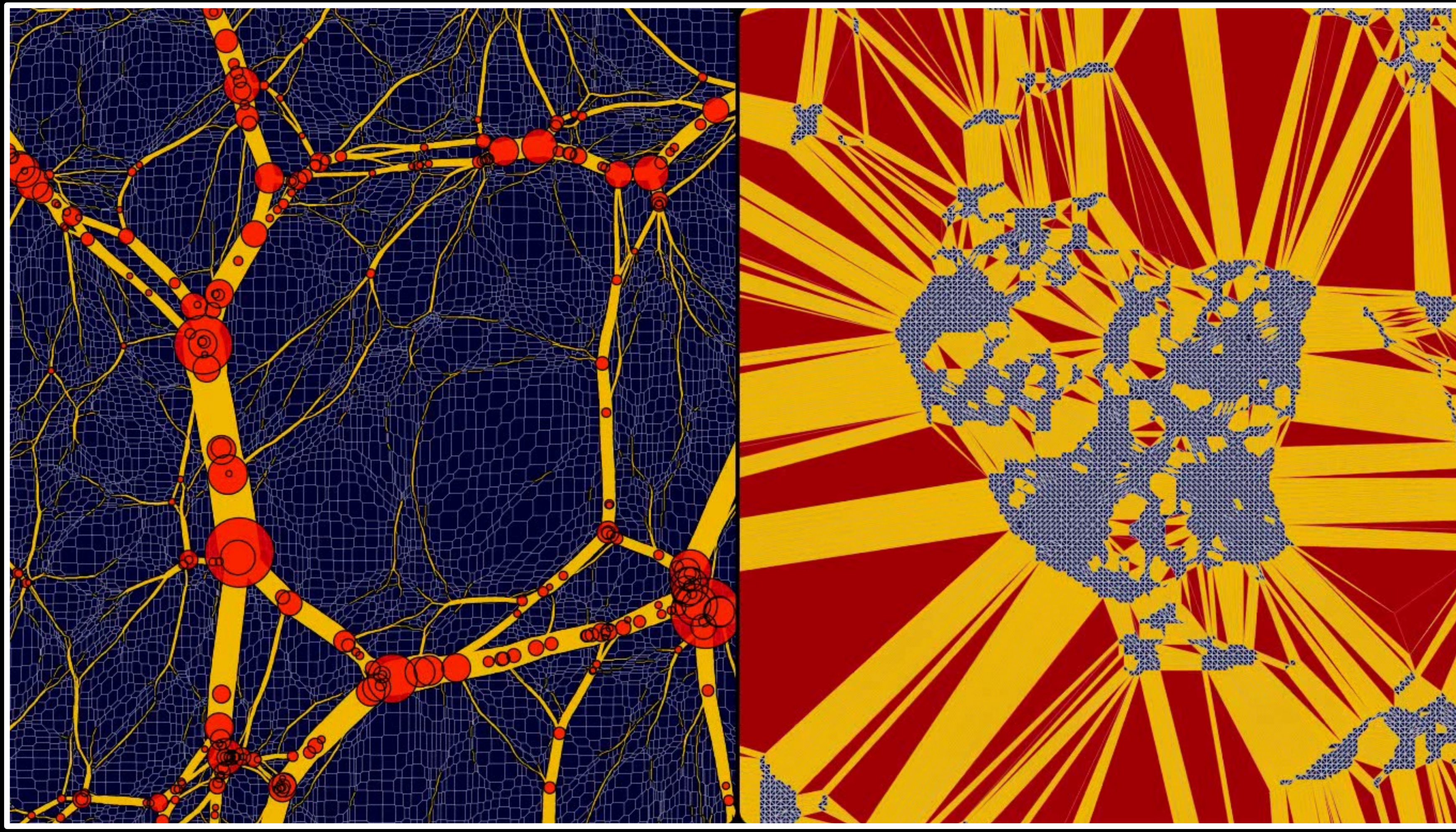
Hidding, vdW et al.

Eulerian – Lagrangian
Voronoi - Delaunay



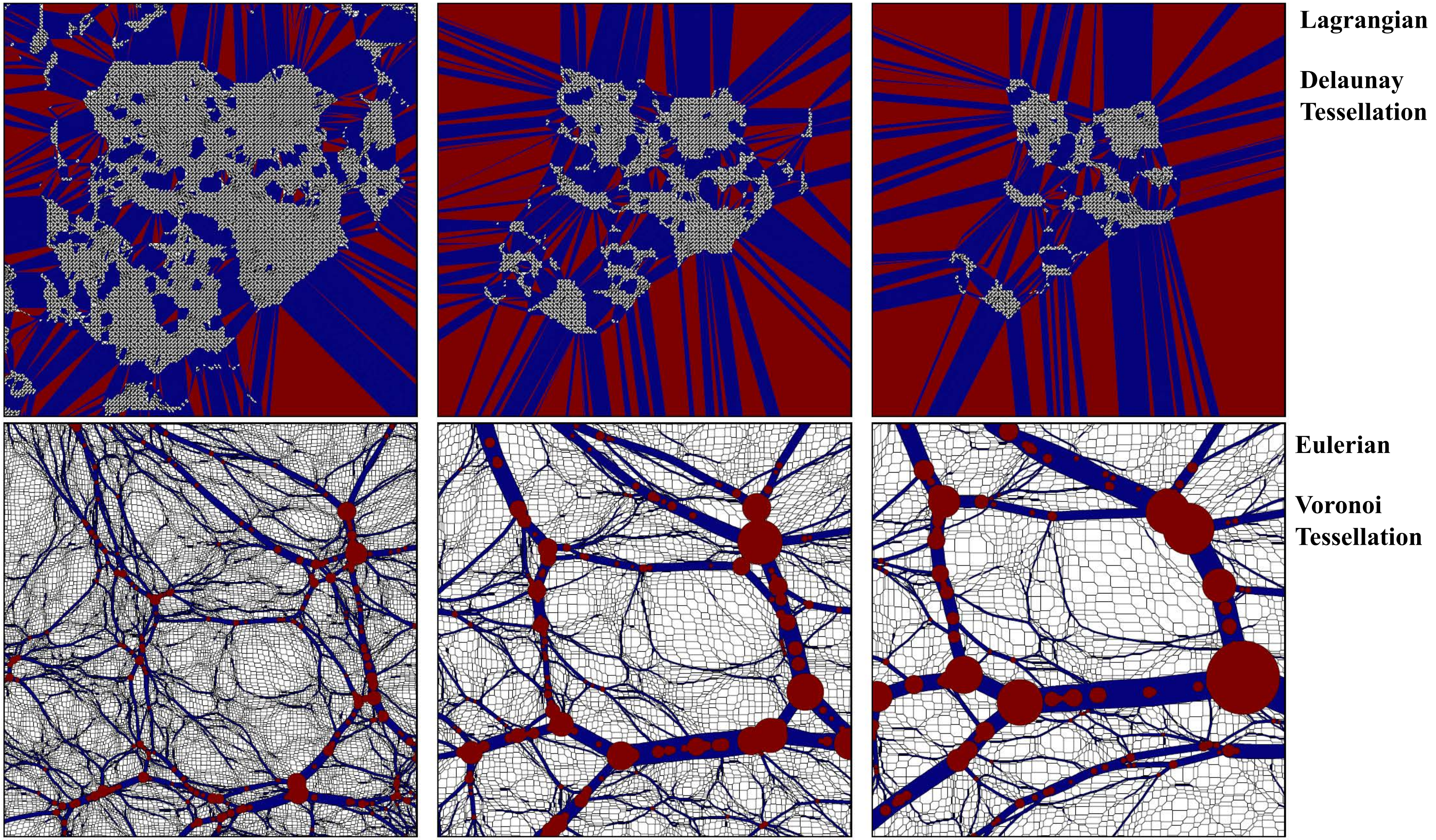


Eulerian – Lagrangian
Voronoi - Delaunay



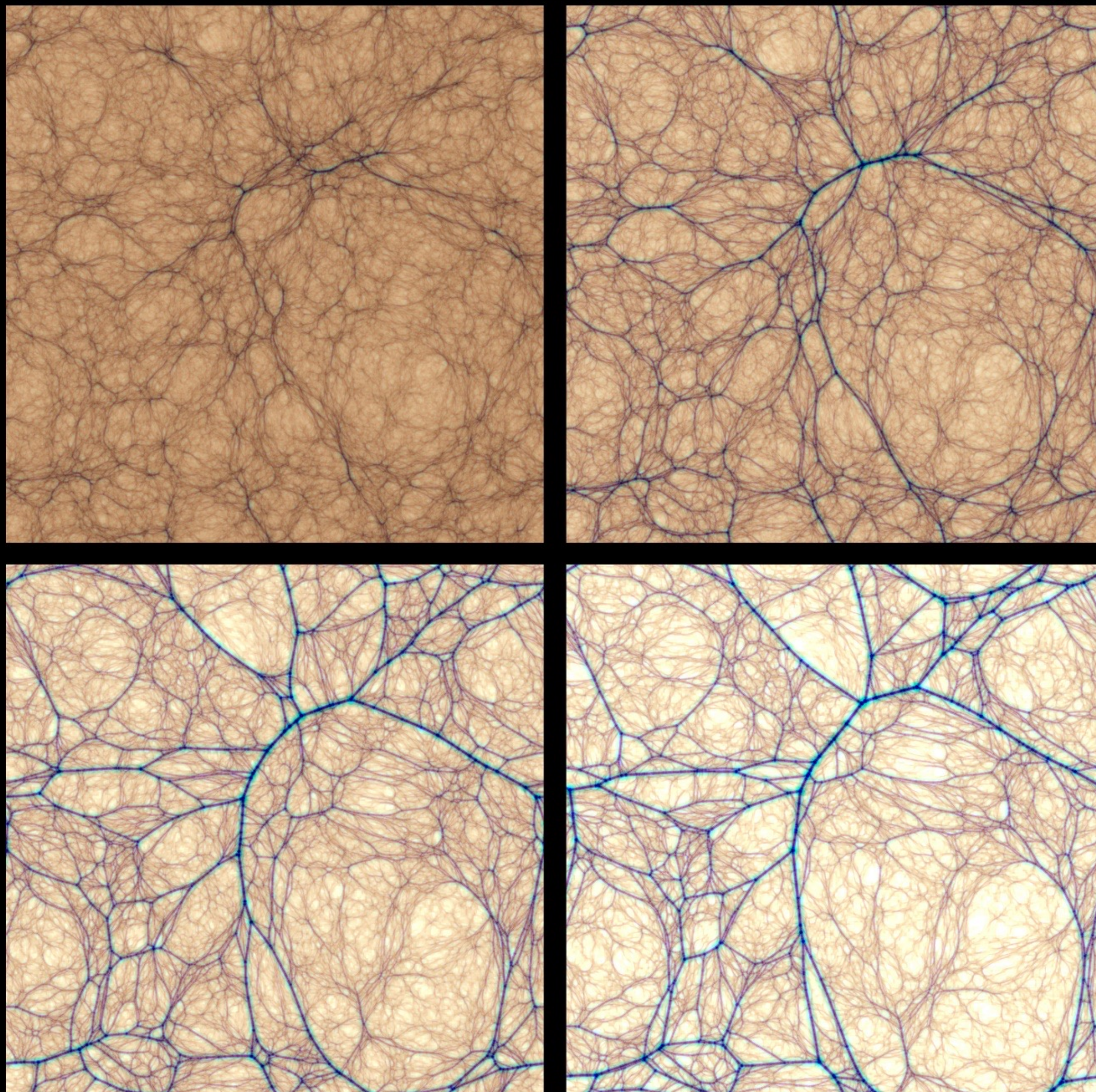
Lagrangian – Eulerian Cosmic Web

Delaunay- Voronoi Tessellations



Hidding, vdW et al.

Hierarachical Evolution

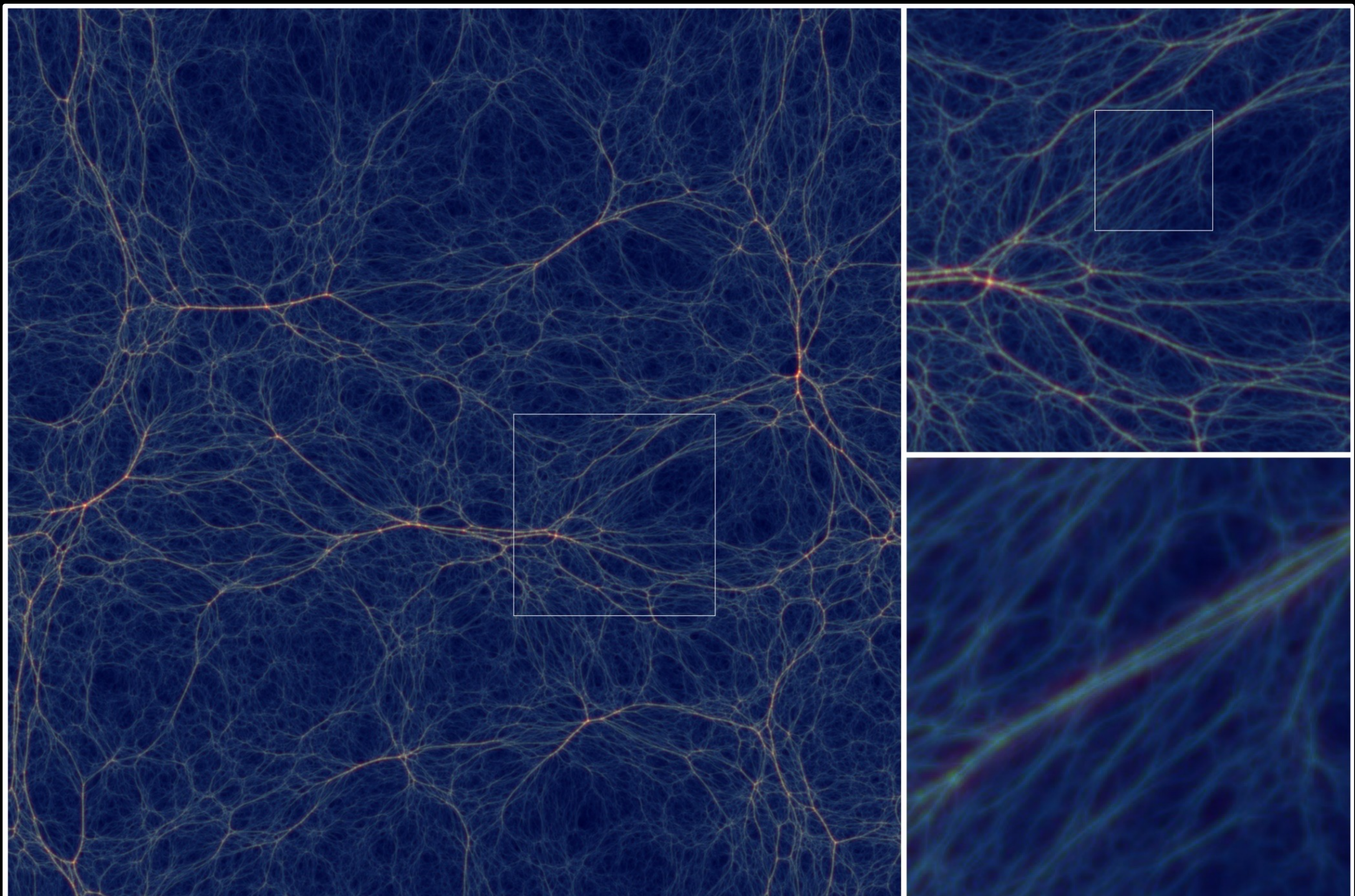


The adhesion formalism is ideal for following the hierarchical buildup of the cosmic web:

- Mathematically:
  as a result of the evolving parabolic curvature of the (velocity) potential, more features get embedded in singular valleys enclosed between potential and convex hull.

- Physically:
  - Clearly visible is the merging of small filaments into ever larger arteries.
  - at the same time, we see the continuous merging of small voids into larger voids, the evolving soapsud of void hierarchy.

Multiscale Structure



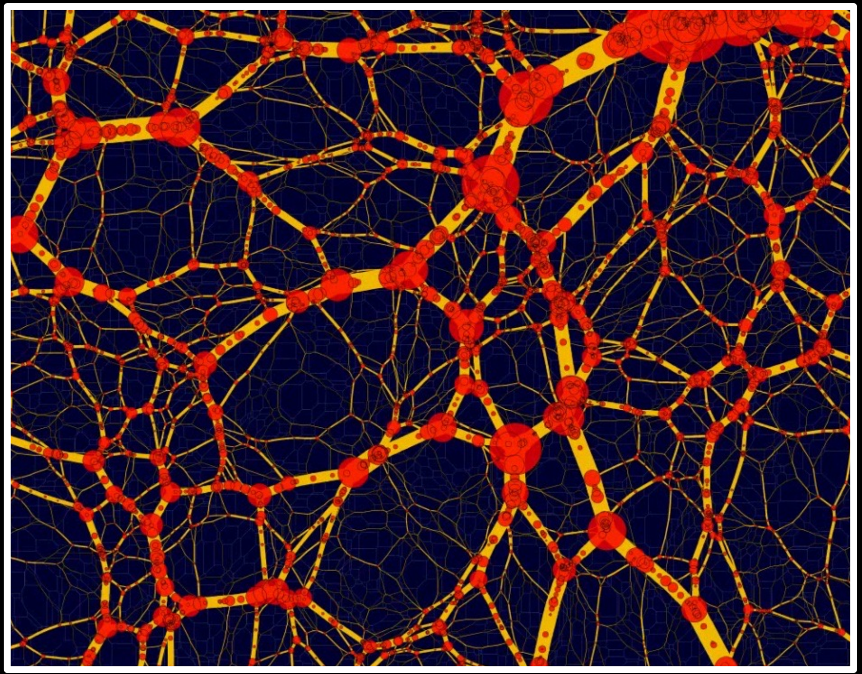
Cosmological Sensitivity Cosmic Web

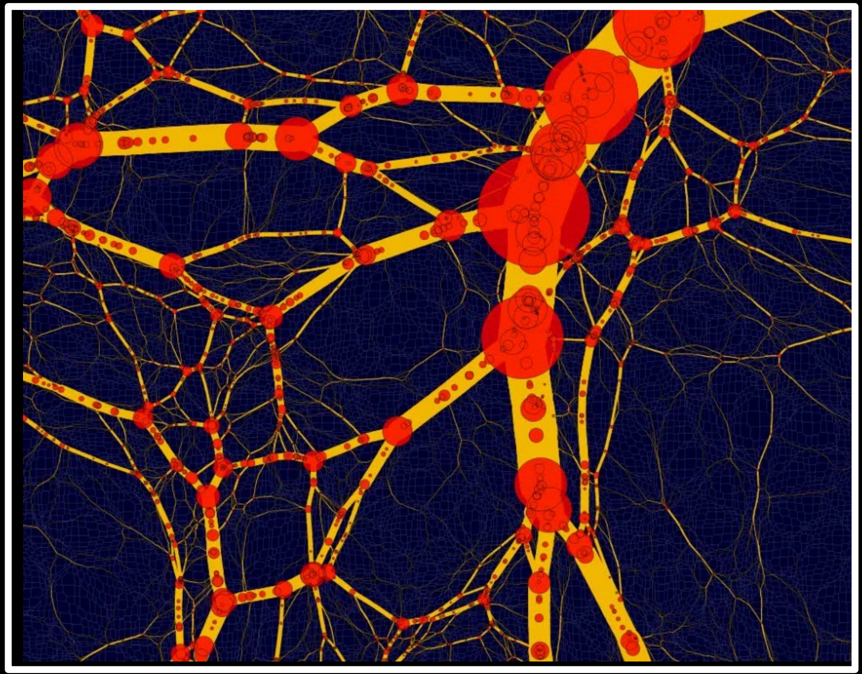
the morphology of the weblike network is
highly sensitive to the underlying cosmology

$P(k) \sim k^{-1.5}$

Hidding 2012/2014

$P(k) \sim k^{-2.0}$

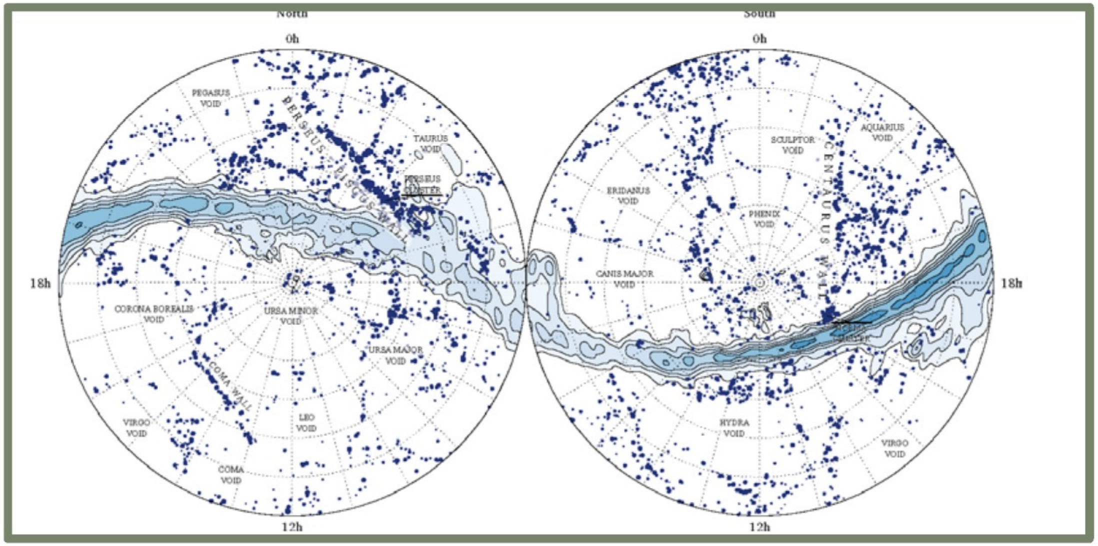




Adhesion Reconstructions :

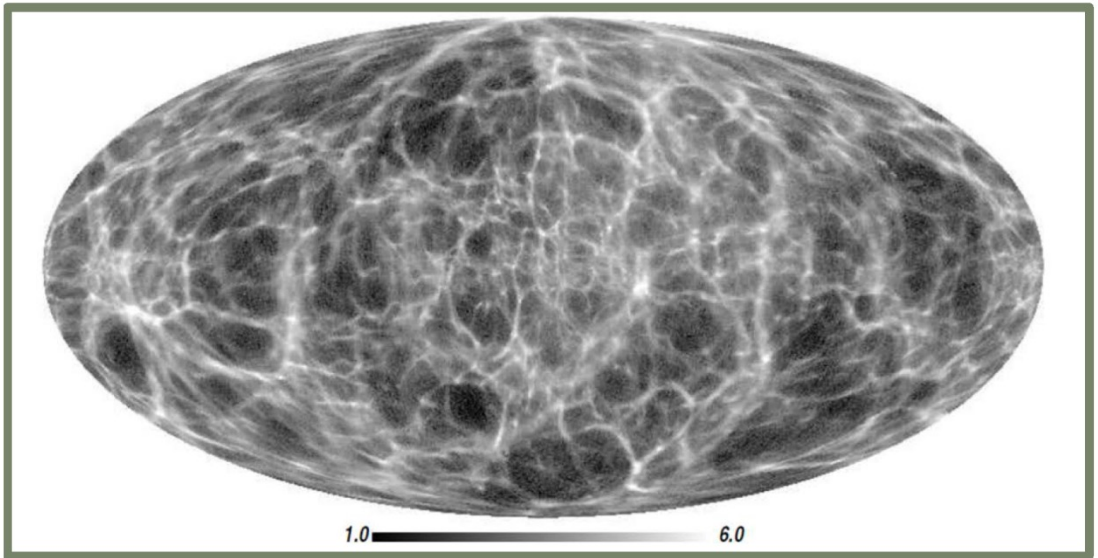
the Cosmic Web in our Local Universe

KIGEN Reconstruction Local Universe …



2MRS survey sky map
Hidding 2015

Depth: ~50 Mpc



KIGEN (Bayesian) reconstruction
Local Cosmic Web:

Kitaura

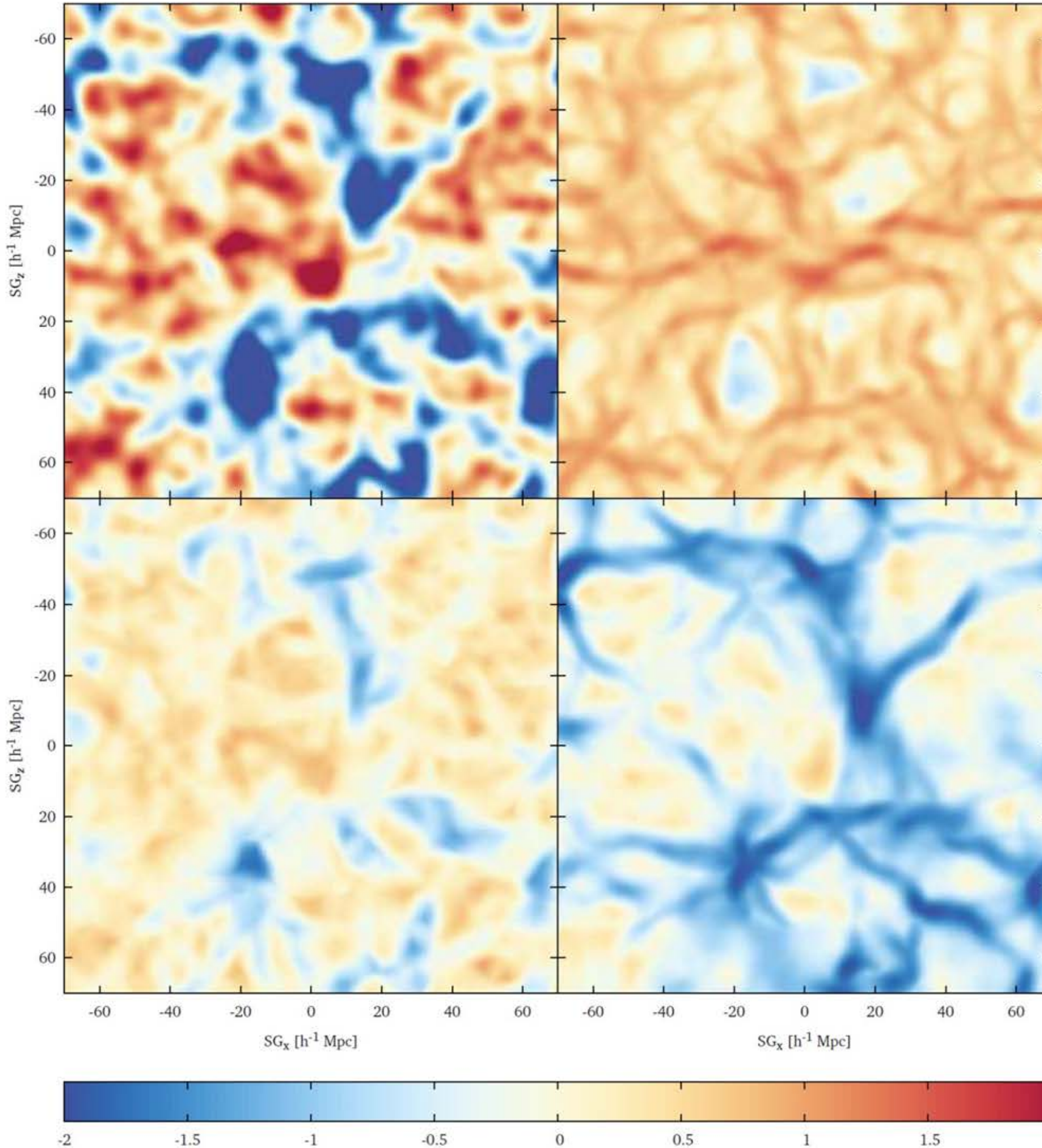
Depth: ~185 Mpc

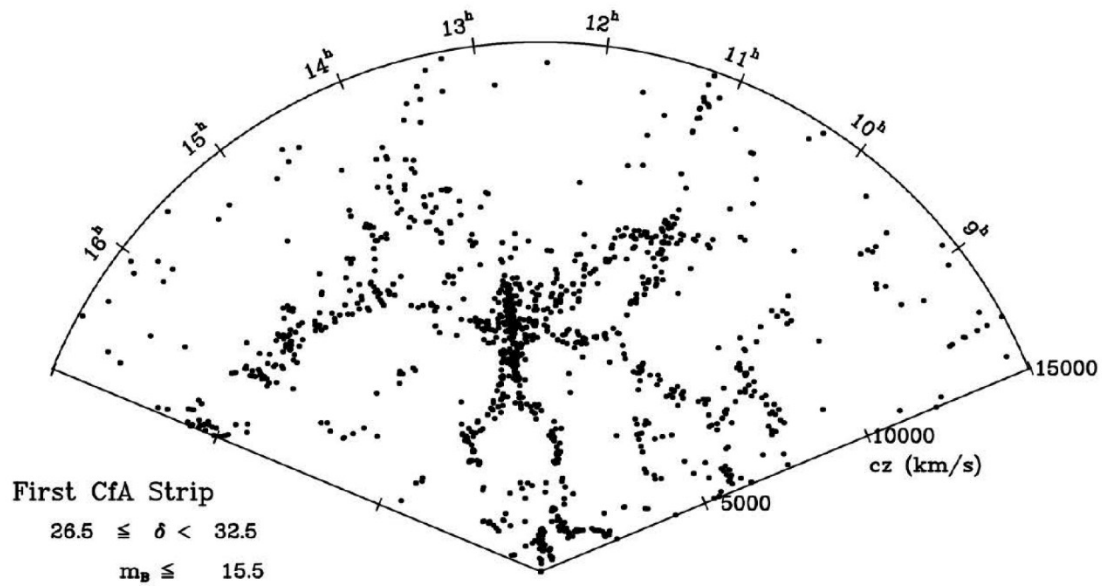
Initial Density & Deformation Field

Local Universe (SG plane)

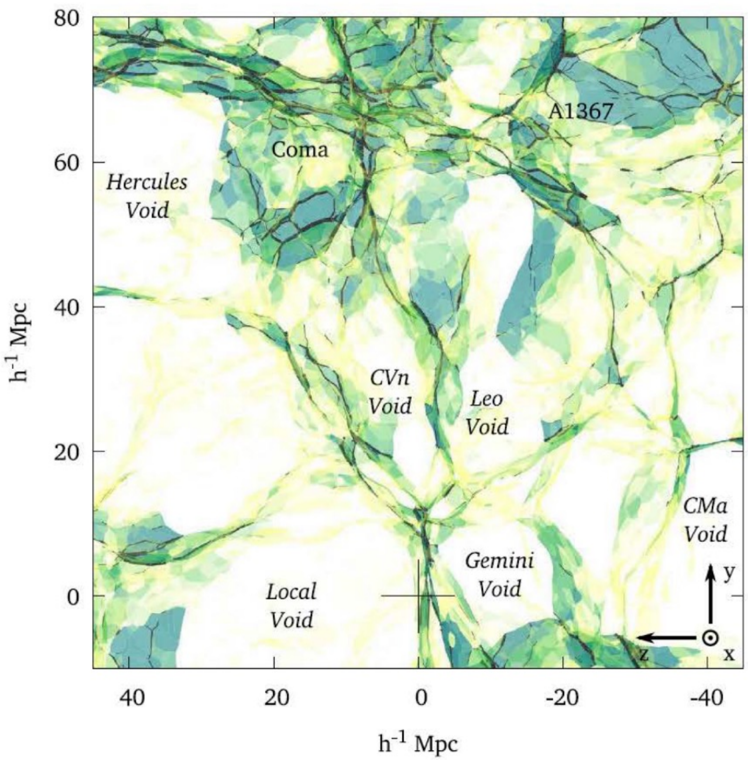
Kitaura & Hess:

25 KIGEN constrained realizations

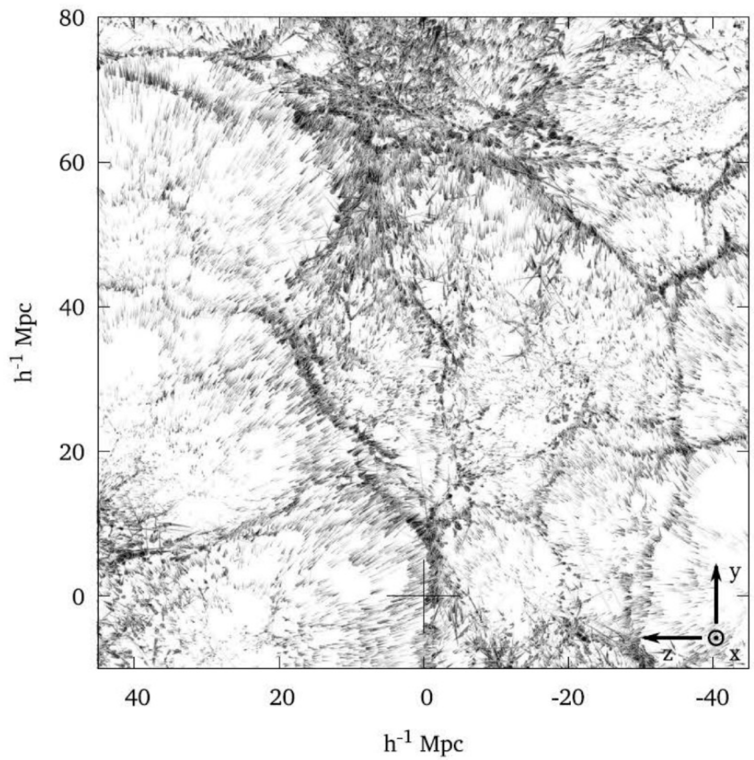




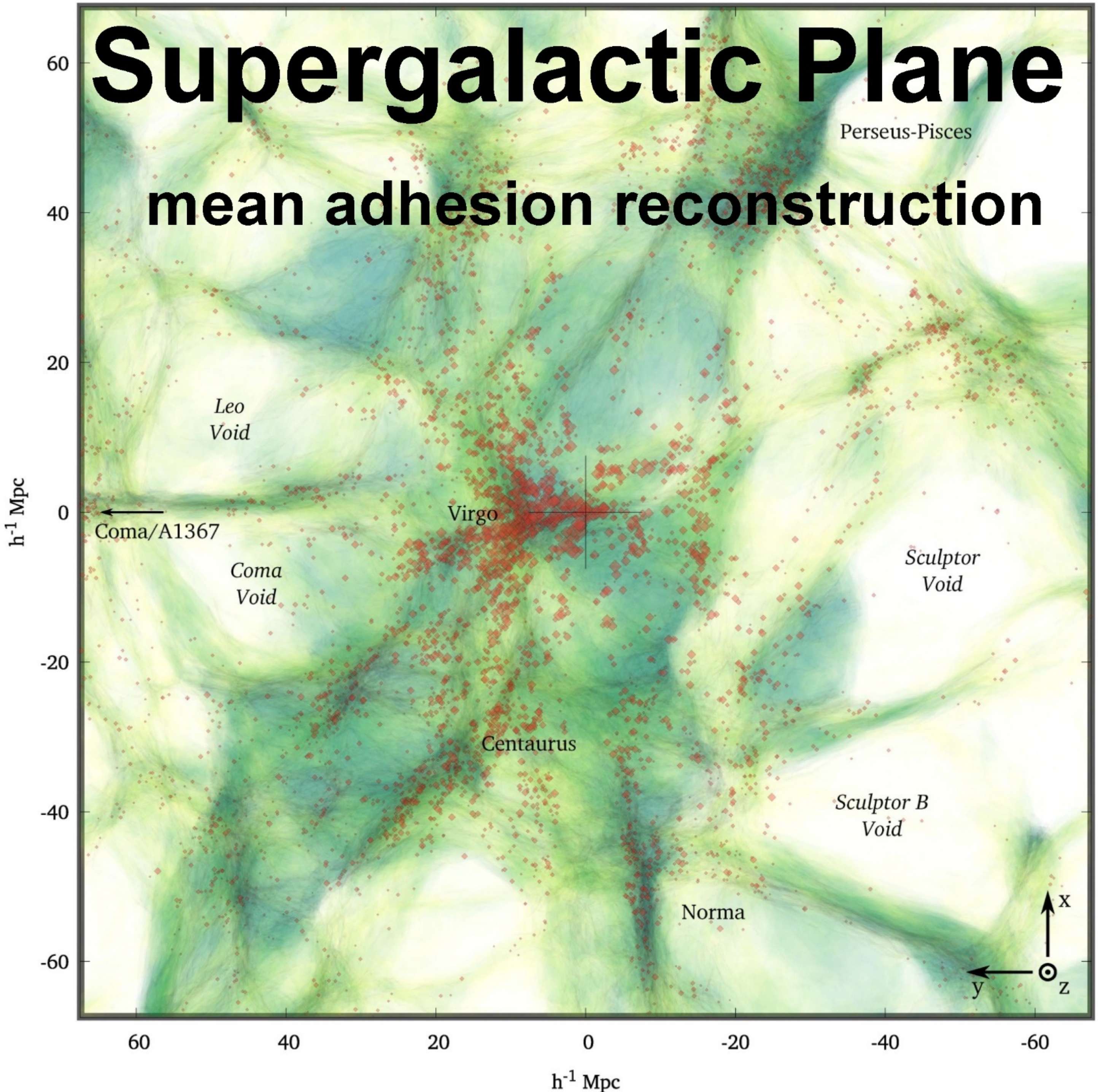
13h
12h
14h
11h
15h
10h
16h
9h
15000
10000
cz (km/s)
5000
First CfA Strip
26.5 ≤ δ < 32.5
mB ≤ 15.5



80
60
40
20
0
h-1 Mpc
A1367
Coma
Hercules Void
CVn Void
Leo Void
CMa Void
Gemini Void
Local Void
y
z
x
40
20
0
-20
-40
h-1 Mpc



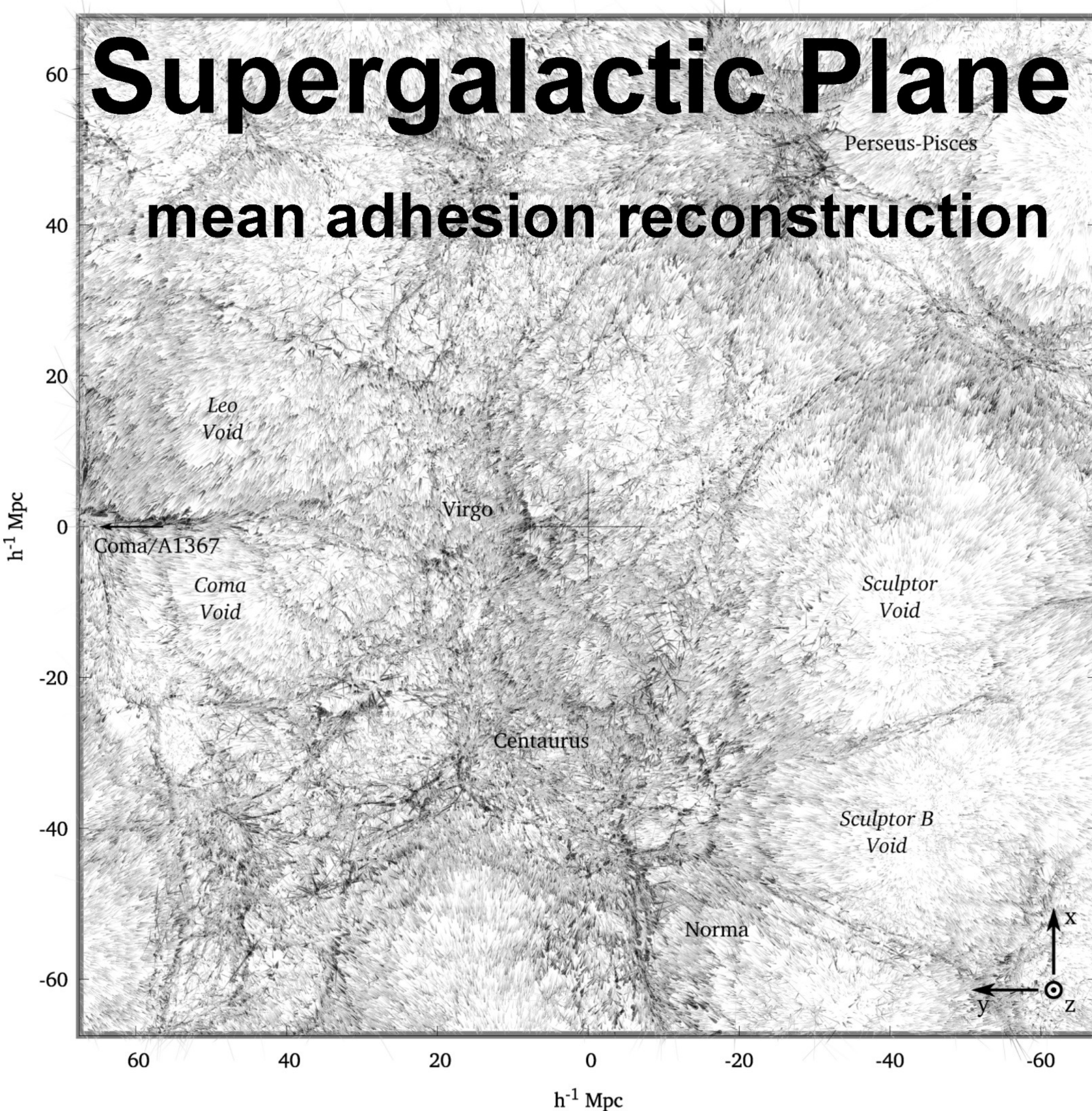
80
60
40
20
0
h-1 Mpc
y
z
x
40
20
0
-20
-40
h-1 Mpc



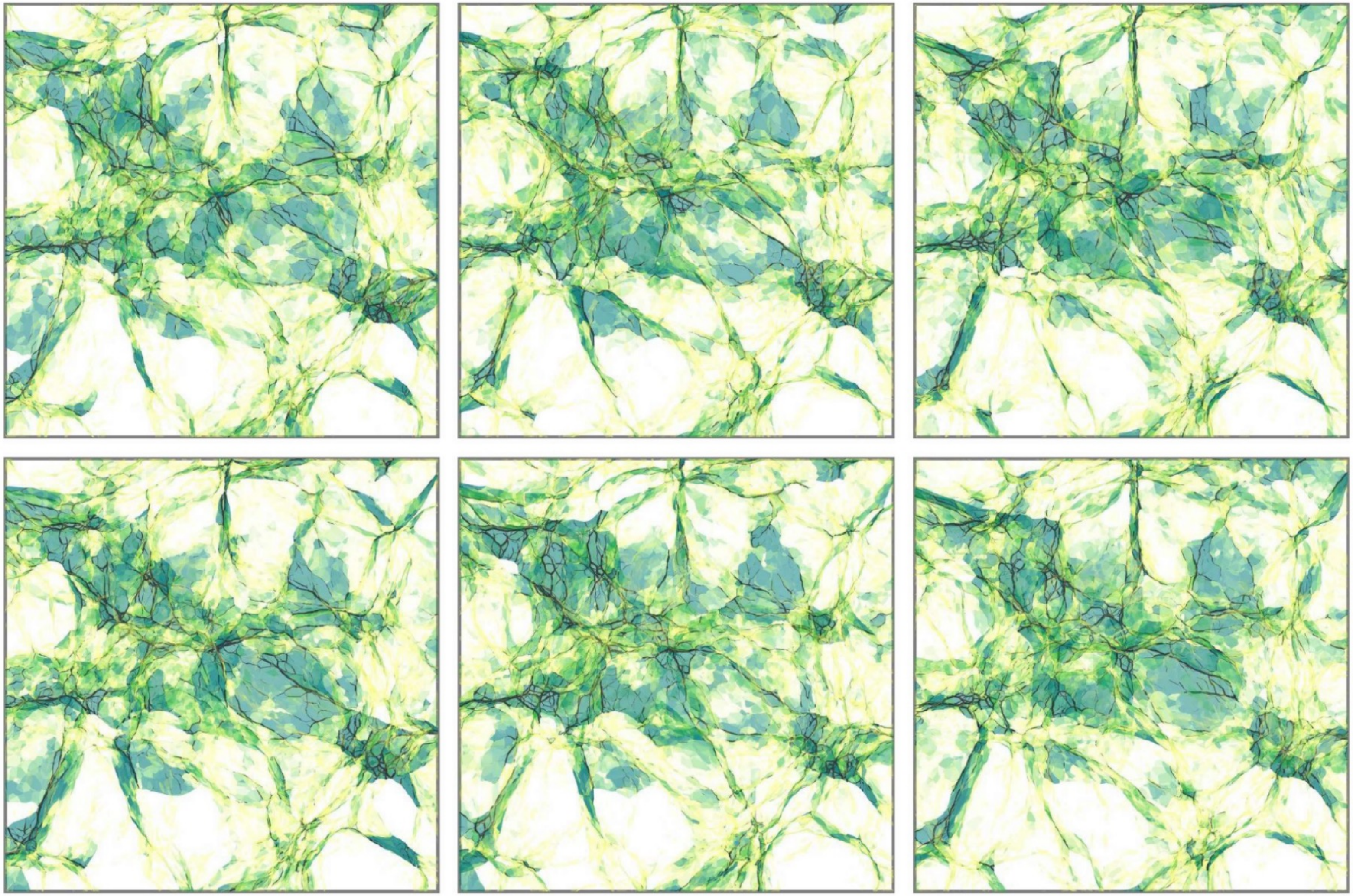
Supergalactic Plane
mean adhesion reconstruction
Perseus-Pisces
Leo
Void
Coma/A1367
Virgo
Coma
Void
Sculptor
Void
Centaurus
Sculptor B
Void
Norma
x
y
z
h-1 Mpc
h-1 Mpc
60
40
20
0
-20
-40
-60

Supergalactic Plane

mean adhesion reconstruction



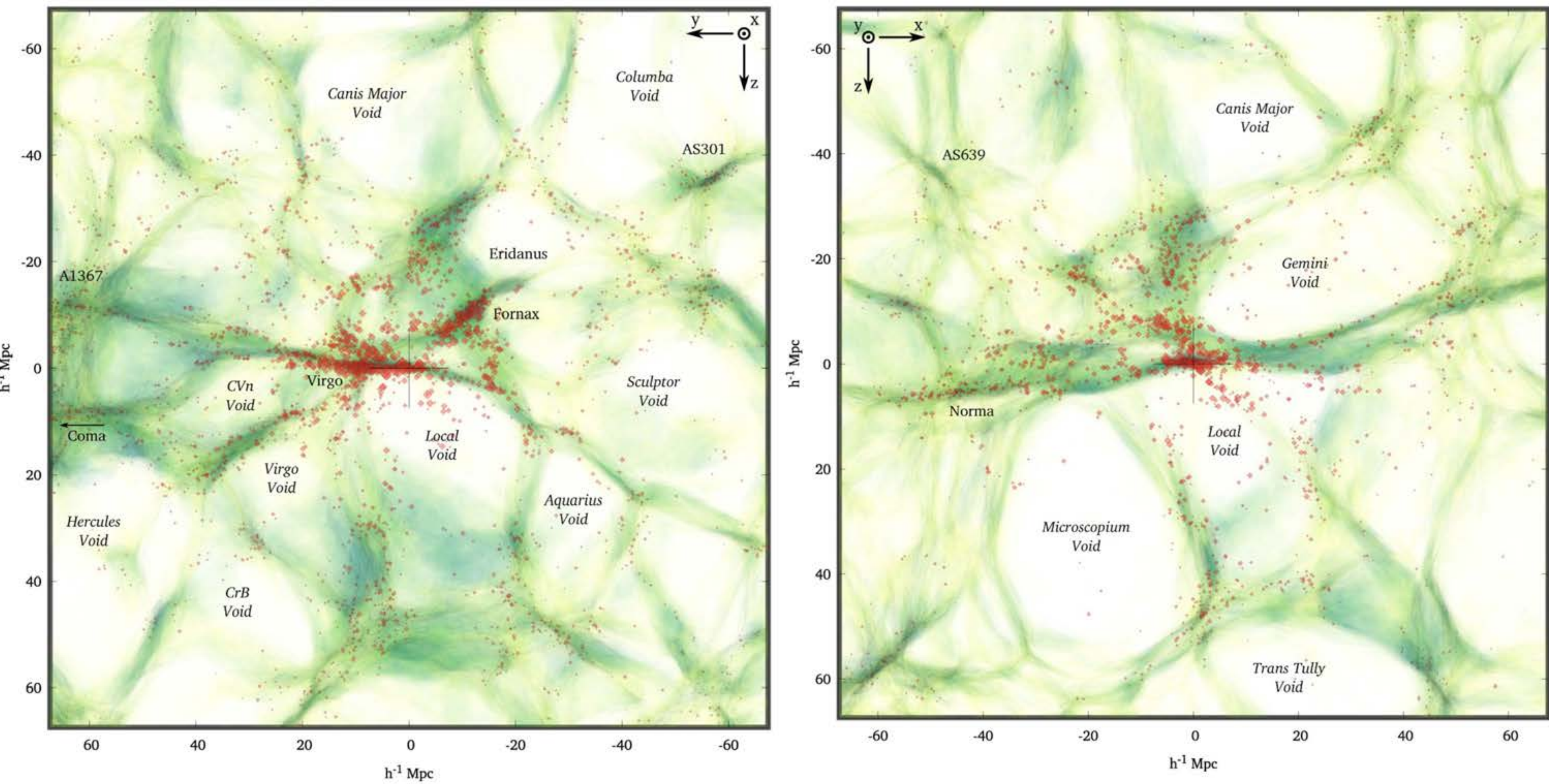
Supergalactic Plane



6 constrained adhesion reconstructions

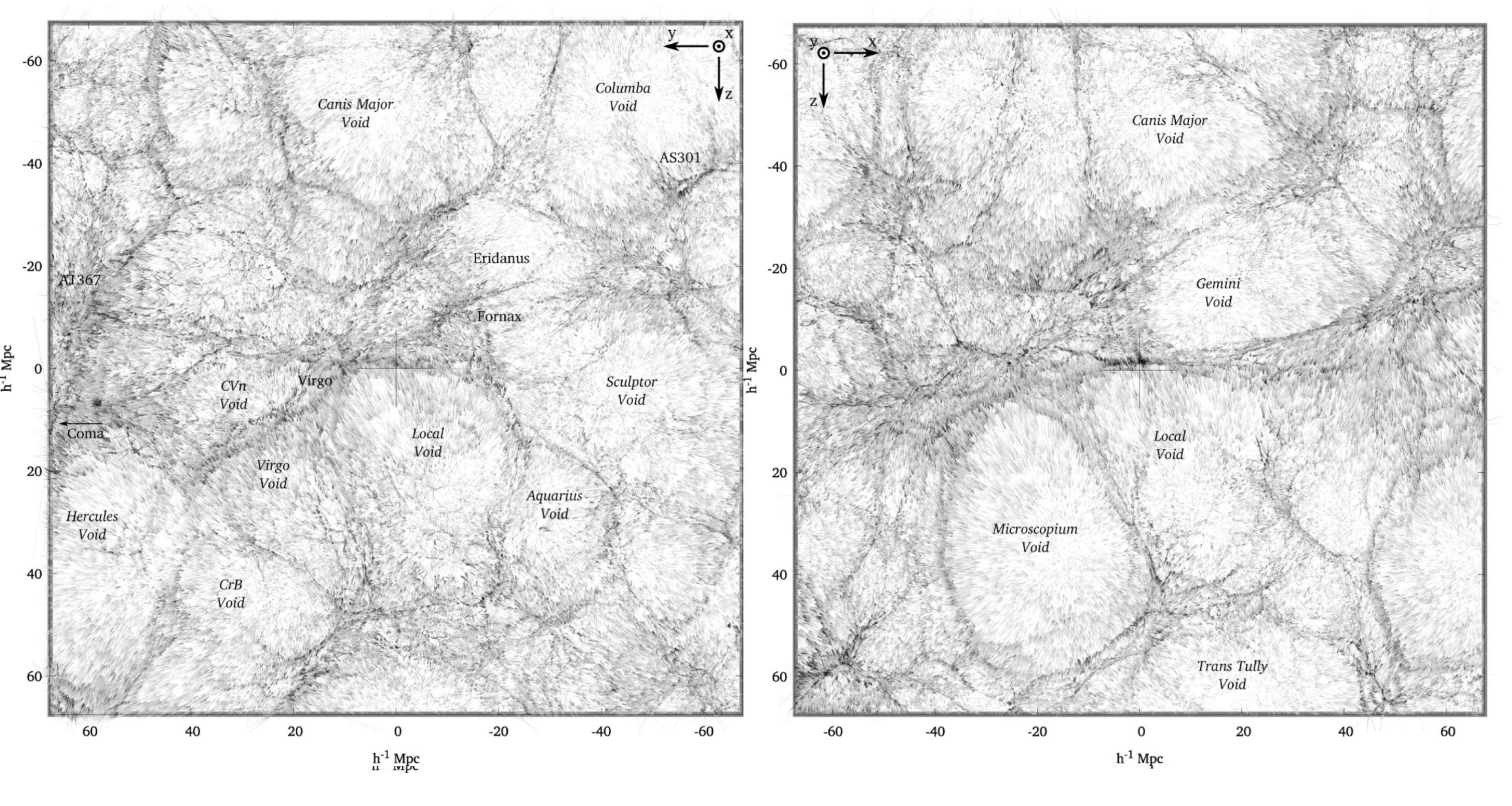
Supergalactic Plane

mean adhesion reconstruction



Supergalactic Plane

mean adhesion reconstruction



Cosmic Web

&

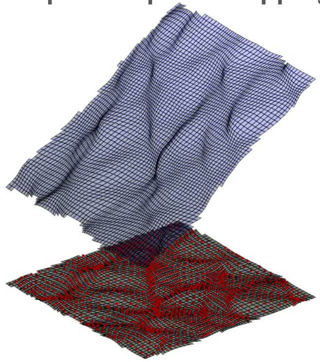
Caustic Skeleton

Cosmic Web – Phase Space Folding

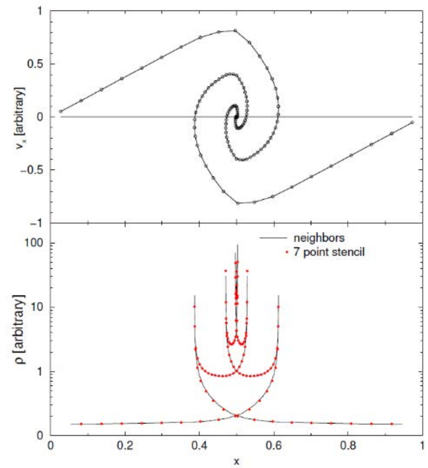
Hidding 2014

Caustic Skeleton & Phase-Space Wrapping

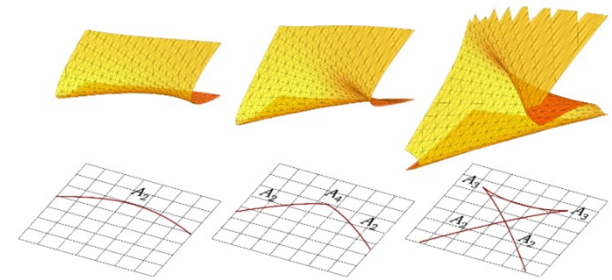
Cosmic Web
phase-space wrapping



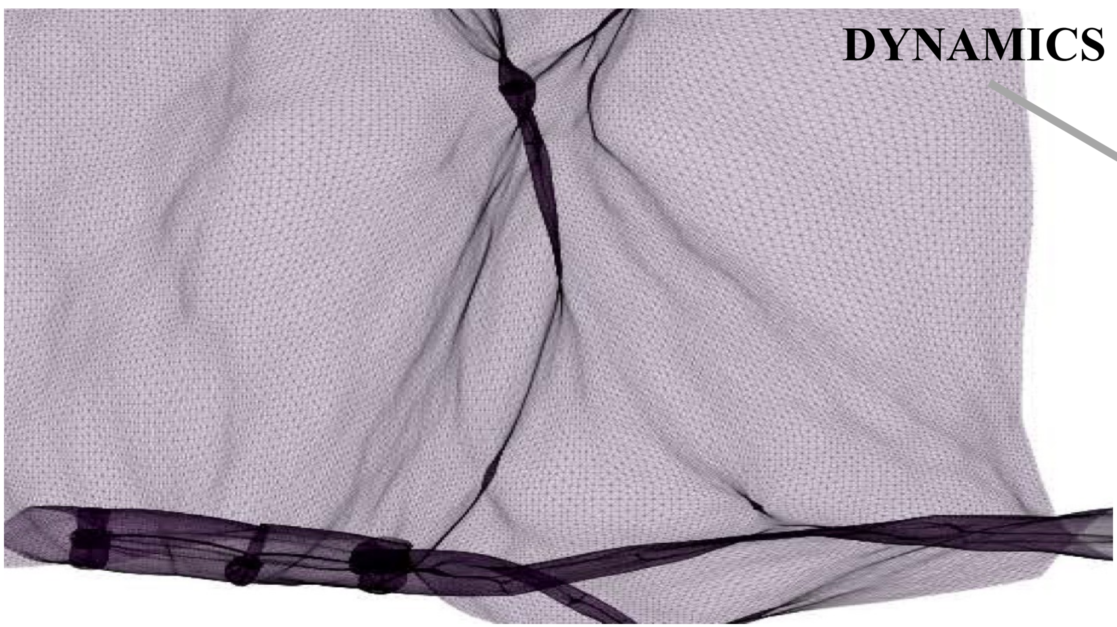
Hidding 2014



Classification phase-space folding –
Structural Morphology



Feldbrugge, vdW et al., JCAP, 2018
Hidding, Shandarin, vdW, MNRAS, 2014



Cosmic Catastrophe Theory:

Lagrangian catastrophe/caustic classification V. Arnold

In Lagrangian space (coordinates q):

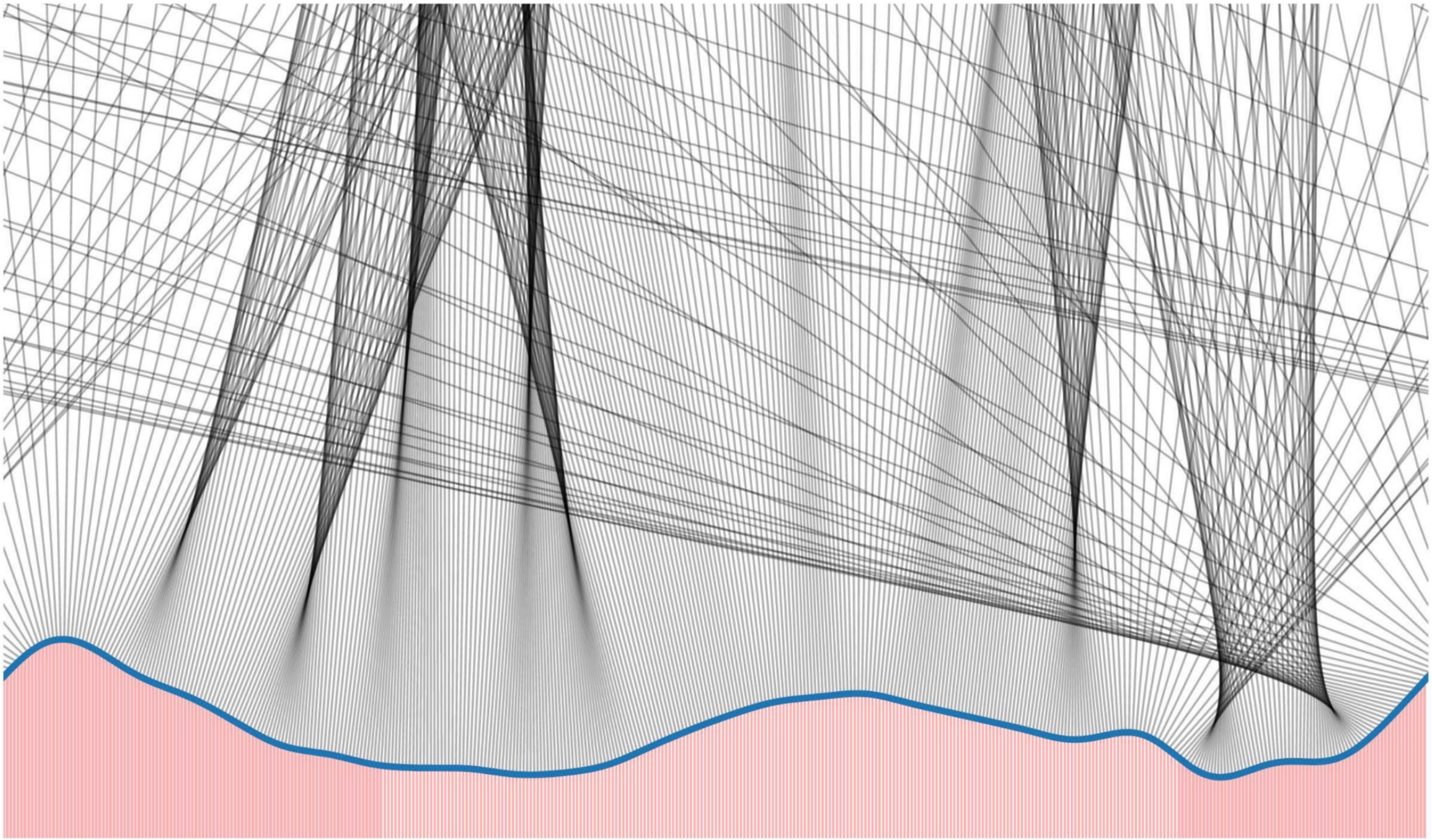
A singularity forms in a manifold M at location q_s when at q_s,

- the deformation tensor eigenvalue $\mu_i(q_s)$
- the corresponding eigenvector $\vec{v}_i\,(q_s)$

when at least one nonzero tangent vector $\vec{T}$

$$\{1+\mu_i(q_s)\}\ \ \vec{v}_i^{\,*}(q_s)\cdot\vec{T}\ = 0$$

Phase Space Dynamics & Tracks



Wilding 2022

Deformation, Streaming & Caustics

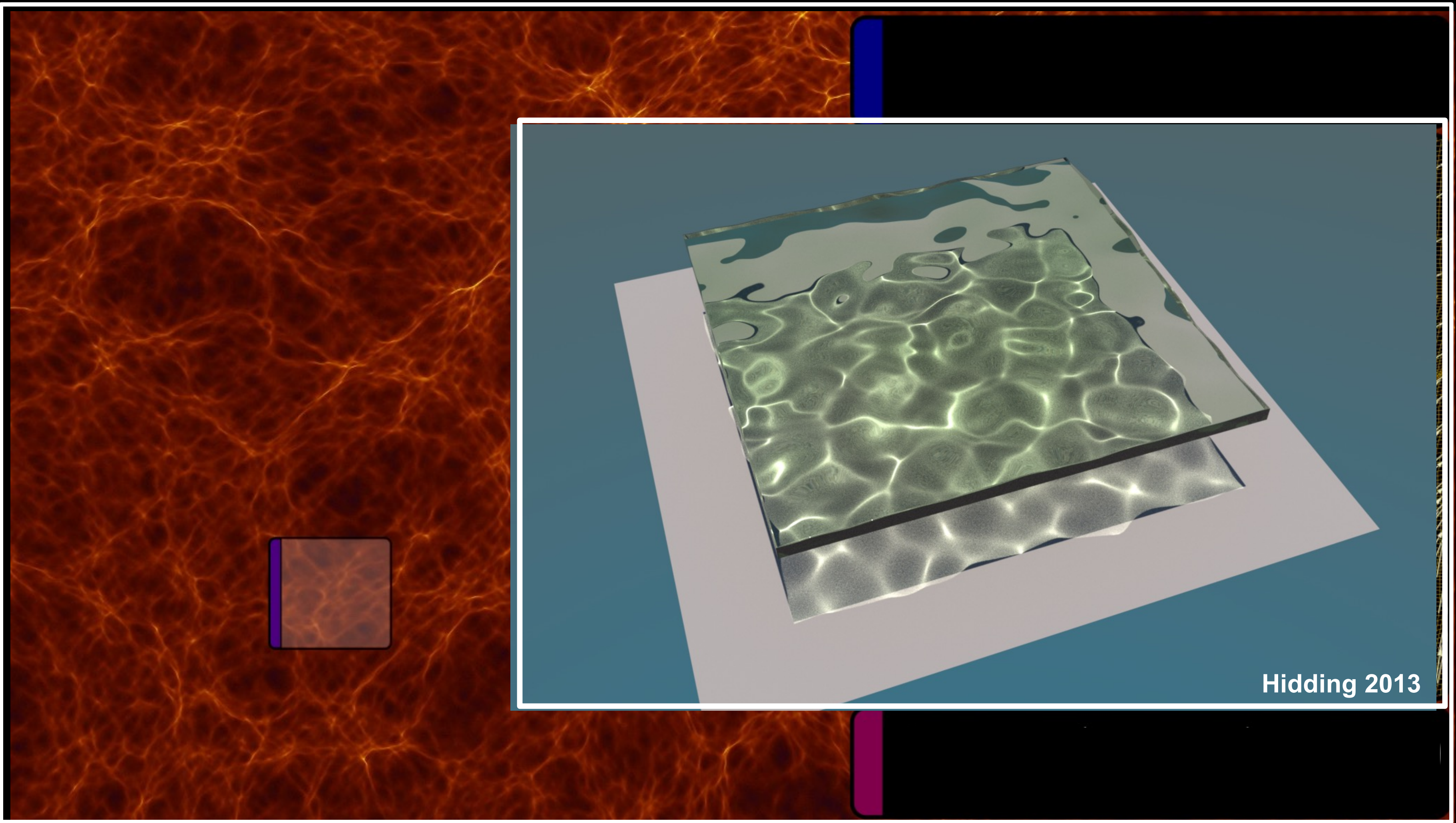
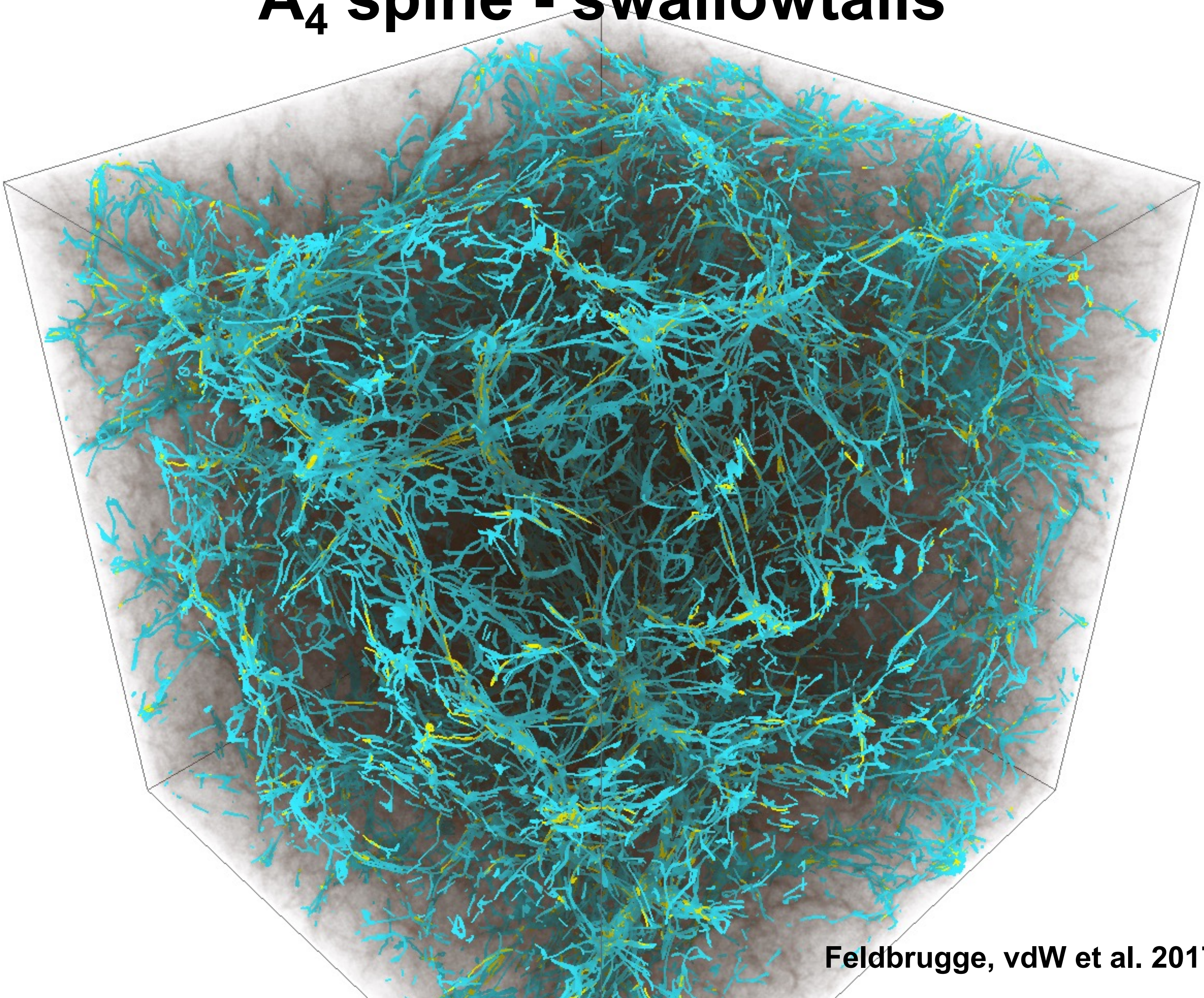


Illustration of the formation of caustics due to streaming paths of light through deforming medium

Skeleton (3D) Cosmic Web:

A_4 spine - swallowtails



Feldbrugge, vdW et al. 2017b