Treeningvõistlus

27.10.2017

1. Leia $n \times n$ maatriksi
$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 0 & 1 \\ & \dots & \dots & \dots & \\ 1 & 0 & \dots & 1 & 1 \\ 0 & 1 & \dots & 1 & 1 \end{pmatrix}$$
pöördmaatriks.

2. Polünoomi
$$P(x) = a_{2018}x^{2018} + a_{2017}x^{2017} + \dots + a_1 x + a_0$$
kordajad rahuldavad võrdust
$$\frac{a_{2018}}{2019} + \frac{a_{2016}}{2017} + \dots + \frac{a_2}{3} + a_0 = 0.$$
Tõesta, et leidub $y \in \mathbb{R}$ nii, et $|y| < 2$ ja $P(y) = 0$.

3. Ruutmaatriks A on *nilpotentne* kui leidub $k \in \mathbb{N}$ nii, et $A^k = 0$. Olgu B ja C nilpotentsed maatriksid, kusjuures $BC = CB$ (st nad kommuteeruvad).
   (a) Tõesta, et $B + C$ on nilpotentne.
   (b) Kas väide kehtib alati isegi siis, kui B ja C ei kommuteeru?

4. Pideva funktsiooni $f : (0, 1) \to \mathbb{R}$ korral
$$x\frac{f'(x)}{f(x)} \leq \frac{1 + x}{1 - x}$$
iga $x \in (0, 1)$ jaoks ja
$$\lim_{x \to 0} \frac{f(x)}{x} = 1.$$
Tõesta, et
$$f(x) \leq x + 2x^2 + 3x^3 + \dots$$
iga $x \in (0, 1)$ jaoks.

5. Leia kõik pidevad funktsioonid $f : (-1, 1) \to \mathbb{R}$ mille puhul
$$f(x + y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}$$
kõikide selliste $x, y \in (-1, 1)$ korral, et $x + y \in (-1, 1)$.

Training competition

27.10.2017

1. Find the inverse of the $n \times n$ matrix
$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 0 & 1 \\ & \dots & \dots & \dots & \\ 1 & 0 & \dots & 1 & 1 \\ 0 & 1 & \dots & 1 & 1 \end{pmatrix}.$$

2. The coefficients of the polynomial
$$P(x) = a_{2018}x^{2018} + a_{2017}x^{2017} + \dots + a_1 x + a_0$$
satisfy
$$\frac{a_{2018}}{2019} + \frac{a_{2016}}{2017} + \dots + \frac{a_2}{3} + a_0 = 0.$$
Prove that $P(y) = 0$ for some $y \in \mathbb{R}$ with $|y| < 2$.

3. An $n \times n$ matrix A is called *nilpotent* if there exists $k \in \mathbb{N}$ such that $A^k = 0$. Let B and C be nilpotent matrices such that $BC = CB$ (i.e., they commute).
   (a) Prove that $B + C$ is nilpotent.
   (b) Does the claim always hold even if B and C do not commute?

4. Let a continuous function $f : (0, 1) \to \mathbb{R}$ satisfy
$$x\frac{f'(x)}{f(x)} \leq \frac{1+x}{1-x}$$
for all $x \in (0, 1)$ and
$$\lim_{x \to 0} \frac{f(x)}{x} = 1.$$
Prove that
$$f(x) \leq x + 2x^2 + 3x^3 + \dots$$
for all $x \in (0, 1)$.

5. Find all continuous functions $f : (-1, 1) \to \mathbb{R}$ such that
$$f(x + y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}$$
for all $x, y \in (-1, 1)$ with $x + y \in (-1, 1)$.

1. Note that $S = A - J$, where S is the given matrix, A is the matrix of 1's, and J is the matrix of 0's with 1's on the perpendicular diagonal. Clearly, $A^2 = nA$, $J^2 = I$, and $JA = AJ = A$. If $M = (m_{ij})$ is the inverse of S, then $MA - MJ = I$. Let m_i denote $\sum_j m_{ij}$. For the first row, this gives

$$m_1 - m_{1k} = 0, \quad k \neq n,$$

and

$$m_1 - m_{1n} = 1,$$

so

$$m_1 = \sum_j m_{1j} = (n-1)m_1 + m_1 - 1$$

and $m_1 = 1/(n-1)$. Hence,

$$M = \frac{1}{n-1}A - J.$$

Indeed,

$$(\frac{1}{n-1}A - J)(A - J) = \frac{1}{n-1}A^2 - JA - \frac{1}{n-1}AJ + J^2 = (\frac{n}{n-1} - 1 - \frac{1}{n-1})A + I = I.$$

2. Note that

$$\int_{-1}^{1} P(x)dx = 2\left(\frac{a_{2018}}{2019} + \frac{a_{2016}}{2017} + \cdots + \frac{a_2}{3} + a_0\right) = 0.$$

So either $P(x) \equiv 0$ or $P(x)$ changes its sign in the interval $(-1, 1)$ and by continuity of P there is a point y in $(-1, 1)$ such that $P(y) = 0$.

3. (a) Let the nilpotency indeces of B and C be b and c, respectively. Since our matrices commute, we get

$$(B + C)^m = \sum_{i=0}^{m} \begin{pmatrix} i \\ m \end{pmatrix} B^i C^{m-i}.$$

Choosing m so that in any summand $i \geq b$ or $m - i \geq c$ (e.g., $m = b + c$ is enough), we get that $(B + C)^m = 0$.

(b) Take

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

4. We have

$$\frac{f'(x)}{f(x)} - \frac{1}{x} \leq \frac{2}{1-x}$$

so that

$$\int_0^x \left(\frac{f'(x)}{f(x)} - \frac{1}{x}\right) dx \leq \int_0^x \frac{2}{1-x} dx,$$

i.e.,

$$\ln \frac{f(x)}{x} \leq \ln \frac{1}{(1-x)^2}.$$

Hence,

$$\frac{f(x)}{x} \leq \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1},$$

as needed.

5. Clearly, $f(0) = 0$. Let $u(x) = \arctan(f(x))$. Then $u(x)$ is continuous and the equation becomes

$$\tan(u(x+y)) = \tan(u(x) + u(y)).$$

So $u(x+y) = u(x) + u(y) + k(x,y)\pi$ with $k(x,y) \in \mathbb{Z}$. But the continuity of u implies that k is continuous and hence constant. So $k \equiv 0$ because $u(0) = 0$.

So we get that $u(x) + u(y) = u(x+y)$ for all $x, y, x+y \in (-1,1)$. This is the Cauchy equation with additional constraints $x, y, x+y \in (-1,1)$. The solution stays the same though, $u(x) = ax$ for suitable $a \in \mathbb{R}$. Hence the answer is $f(x) = \tan(ax)$, where $|a| \leq \pi/2$ to keep f continuous.

How to solve Cauchy equation: first note that $u(nx) = nu(x)$ for all $n \in \mathbb{N}$, $x, nx \in (-1,1)$. Now $u(x/n) = u(x)/n$ for all $n \in \mathbb{N}$, $x \in (-1,1)$. Then $u((m/n)x) = mu(x/n) = (m/n)u(x)$ forall $m, n \in \mathbb{N}$, $x \in (-1,1)$. Also, clearly $u(-x) = -u(x)$ for all $x \in (-1,1)$. Hence $u(qx) = qu(x)$ for all $q \in \mathbb{Q}$, $x, qx \in (-1,1)$. Take $r \in \mathbb{R}$ such that $x, rx \in (-1,1)$. There exists a sequence $(q_n) \subset \mathbb{Q}$ such that $q_n \to r$ and $|q_n| \leq r$. By continuity of u, one has $q_n u(x) = u(q_n x) \to u(rx)$. On the other hand, $q_n u(x) \to ru(x)$. So $u(rx) = ru(x)$ for all $r \in \mathbb{R}$, $x, rx \in (-1,1)$. Denoting $a := u(x_0)/x_0$ for some $x_0 \in (-1,1)$, we get that $u(x) = u(x_0 \cdot x/x_0) = (x/x_0)u(x_0) = ax$ for all $x \in (-1,1)$.

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