Treeningvõistlus

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1. (a) Leia kõik sellised pidevad funktsioonid $f : [0, \infty) \to \mathbb{R}$, et iga $x \geq 0$ korral
$$f(x^2) = f(x).$$
   (b) Kas leidub lõpmata palju erinevaid pidevaid funktsioone $f : (1, \infty) \to \mathbb{R}$, mille jaoks iga $x > 1$ korral
$$f(x^2) - f(x) = \ln 2?$$

2. Olgu A ja B reaalsed $n \times n$ maatriksid. Tõesta, et
$$A(A+B)^{-1}B = B(A+B)^{-1}A$$
eeldusel, et
   (a) maatriksid $A + B$, A ja B on pööratavad,
   (b) maatriks $A + B$ on pööratav.

3. Iga $n \in \mathbb{N}$ korral tähistame
$$S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}.$$
Tõesta, et
$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$
ja leia
$$\lim_{n \to \infty} S_n.$$

4. Milliste $m \geq 1$ korral leiduvad täisarvud $n_0 < n_1 < \ldots < n_m$ ja täisarvuliste kordajatega polünoom P, mille aste on m, nii, et
$$|P(n_0)| = |P(n_1)| = \ldots = |P(n_m)| = 1?$$

Solutions

1. (a) These are constant functions. They clearly satisfy the condition. On the other hand, fix $x > 0$ and denote $x_n := x^{\frac{1}{2^n}}$. Then $x_n \to 1$. So $f(x_n) \to f(1)$. But

$$f(x) = f(x_0) = f(x_1) = \ldots = f(x_n),$$

so $f(x) = f(1)$. Then, by continuity, also $f(0) = f(1)$.

   (b) Yes, because $h(x) = \ln\ln x$ as well as $h(x) + C$, where $C \in \mathbb{R}$ is a constant, satisfy the condition.

2. (a) We have

$$\begin{aligned} A(A+B)^{-1}B &= (A^{-1})^{-1}(A+B)^{-1}(B^{-1})^{-1} \\ &= (B^{-1}(A+B)A^{-1})^{-1} \\ &= (B^{-1} + A^{-1})^{-1}. \end{aligned}$$

Since the latter expression is symmetric with respect to A and B, we get the claim.

   (b) This is because

$$A(A+B)^{-1}B + B(A+B)^{-1}B = (A+B)(A+B)^{-1}B = B$$

and

$$B(A+B)^{-1}A + B(A+B)^{-1}B = B(A+B)^{-1}(A+B) = B.$$

3. Set

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Then

$$S_n = H_{2n} - 2(\frac{1}{2}H_n) = H_{2n} - H_n.$$

To find a limit, observe that

$$\lim_n S_n = \lim_n \frac{1}{n}\sum_{i=1}^{n} \frac{1}{1+\frac{i}{n}} = \int_0^1 \frac{dx}{1+x} = \ln 2.$$

4. For $m = 1$: $P(x) = x$, $\{n_i\} = \{-1, 1\}$.
For $m = 2$: $P(x) = x(x-1) - 1$, $\{n_i\} = \{-1, 0, 1\}$.
For $m = 3$: $P(x) = (x+1)x(x-2) + 1$, $\{n_i\} = \{-1, 0, 1, 2\}$.
There are no such polynomials for $m \geq 4$. Using the Lagrange interpolation, we can show that

$$\max |P(n_i)| \geq |a_m| \frac{m!}{2^m} \geq \frac{m!}{2^m} \geq \frac{4!}{2^4} = \frac{24}{16} > 1$$

where a_m is the coefficient at x^m. Indeed, if we denote

$$f(x) = (x - n_0)(x - n_1)(x - n_2)\ldots(x - n_m),$$

then the Lagrange interpolation formula gives

$$P(x) = \sum_{i=0}^{m} \frac{P(n_i)}{f'(n_i)} \frac{f(x)}{x - n_i}.$$

Looking at the coefficient at x_m we obtain

$$a_m = \sum_{i=0}^{m} \frac{P(n_i)}{f'(n_i)}$$

and

$$|a_m| \leq \max |P(n_i)| \sum_{i=0}^{m} \frac{1}{|f'(n_i)|}.$$

Note that

$$|f'(n_i)| = |(n_i - n_0)(n_i - n_1) \dots (n_i - n_{i-1})(n_i - n_{i+1}) \dots (n_i - n_m)| \geq i!(m-i)!.$$

Hence

$$\sum_{i=0}^{m} \frac{1}{|f'(n_i)|} \leq \frac{1}{m!} \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} = \frac{2^m}{m!}.$$