Tartu Ülikooli üliõpilaste matemaatikaolümpiaad

15.05.2015

1. Olgu $f : \mathbb{R} \to \mathbb{R}$. Tõesta, et järgmised väited on samaväärsed:

   (a) f on diferentseeruv ja f' on perioodiline;

   (b) $f(x) = g(x) + c \cdot x$ iga $x \in \mathbb{R}$ korral, kus $c \in \mathbb{R}$ ning $g : \mathbb{R} \to \mathbb{R}$ on diferentseeruv ja perioodiline.

   Info: funktsioon $h : \mathbb{R} \to \mathbb{R}$ on *perioodiline*, kui leidub $p > 0$ nii, et $h(x + p) = h(x)$ iga $x \in \mathbb{R}$ korral.

2. Milliste positiivsete täisarvude n korral kehtib järgmine väide? Kui P on reaalsete kordajatega n-muutuja polünoom ning $P(x) > 0$ iga $x \in \mathbb{R}^n$ korral, siis leidub $\varepsilon > 0$ nii, et $P(x) > \varepsilon$ iga $x \in \mathbb{R}^n$ korral.

3. Leia kõik funktsioonid $f : \mathbb{R} \to \mathbb{R}$, mille puhul

   $$f(x + f(x) + f(y)) = f(y + f(x)) + x + f(y) - f(f(y))$$

   kõikide x, $y \in \mathbb{R}$ korral.

4. Olgu n positiivne täisarv. Leia kõik maatriksid $A \in \mathrm{Mat}_n(\mathbb{R})$, mille puhul

   $$A^2B + BA^2 = 2ABA$$

   kõikide $B \in \mathrm{Mat}_n(\mathbb{R})$ korral.

5. Kehtigu maatriksite A, $B \in \mathrm{Mat}_3(\mathbb{Z})$ ja mingi positiivse täisarvu k korral võrdus

   $$AB = \begin{pmatrix} 1 & 2k & k(2k+1) \\ 0 & 1 & 2k \\ 0 & 0 & 1 \end{pmatrix}.$$

   Tõesta, et leidub $C \in \mathrm{Mat}_3(\mathbb{Z})$ nii, et $BA = C^k$.

6. Olgu antud täisarv $n \geq 2$. Kas sümmeetrilises rühmas S_n on rohkem elemente paaris- või paaritu järguga?

   Info ülesande lahenduseks:

   (a) Rühma elemendi *a järk* on vähim positiivne täisarv k nii, et $a^k = 1$ (siin 1 on rühma ühikelement) või lõpmatus, kui sellist arvu ei leidu. Lõplikus rühmas igal elemendil on lõplik järk.

   (b) *Sümmeetriline rühm* S_n koosneb kõikidest hulga $\{1, 2, \ldots, n\}$ *permutatsioonidest* ehk selle hulga bijektiivsetest teisendustest, kusjuures rühma korrutamine on defineeritud kui teisenduste kompositsioon ehk järjest rakendamine. Nii et kahe permutatsiooni $\sigma_1, \sigma_2 \in S_n$ puhul $(\sigma_1\sigma_2)(x) = \sigma_1(\sigma_2(x))$ iga $x \in \{1, \ldots, n\}$ korral. Selle tehte suhtes S_n on tõepoolest rühm (ehk tehe on assotsiatiivne, leidub ühikelement (tähistatakse *id*), ning igal elemendil leidub pöördelement).

(c) Konkreetset permutatsiooni võib tähistada näiteks Cauchy kahe rea notatsiooniga

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}.$$

Teine võimalus on esitada sõltumate tsüklite korrutisena. Näiteks,

$$(15)(34) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 3 & 1 \end{pmatrix} \quad \text{ja} \quad (125)(34) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}.$$

(d) Permutatsiooni, kus ainult kaks arvu on omavahel vahetatud, nimetatakse *transpositsiooniks*. Ta on esitatav kujul (kl). Iga permutatsioon on esitatav (mitmel viisil) transpositsioonide korrutisena, kusjuures ühe permutatsiooni kõikides sellistes esitustes on transpositsioonide arvud sama paarsusega. Selle invariantse paarsuse järgi nimetatakse permutatsiooni kas paaris- või paarituks. NB! Permutatsiooni enda paarsus ja permutatsiooni järgu paarsus on erinevad mõisted.

(e) Rühmas S_n on paarispermutatsioone täpselt sama palju kui paarituid.

Solutions

1. (b) $\Rightarrow$ (a) is clear because $f'(x) = g'(x) + c$.

   (a) $\Rightarrow$ (b). If $f'(x+p) = f'(x)$ for all x, then $(f(x+p) - f(x))' = 0$ and there exists a constant $d \in \mathbb{R}$ such that $f(x+p) - f(x) = d$ for all x. Defining g by $f(x) = g(x) + cx$ for any c, we see that g is differentiable and $g(x+p) - g(x) + cp = d$ for all x. So g is periodic whenever $c = \dfrac{d}{p}$.

2. *Answer:* $n = 1$.

   Solution: Case $n = 1$. Since P is everywhere positive, it has an even degree. So $\lim_{x\to\pm\infty} P(x) = \infty$ and there exists $M > 0$ such that $P(x) > 1$ whenever $|x| > M$. On the other hand, P is a continous function and attains its minimum $P(x_0)$ on the compact set $[-M, M]$. But $P(x_0) > 0$ by the assumption, so $\varepsilon = \min(1, P(x_0))$ has the required property.

   Case $n \geq 2$. Consider the polynomial $P(x, y, \dots) = x^2 + (xy-1)^2$. It is everywhere positive, since the summands cannot be both zero at the same time. On the other hand, for $x > 0$ we have

   $$P(x, \frac{1}{x}, \dots) = x^2 \to_{x\to 0} 0.$$

3. Let us first show that f is injective. Assume $f(a) = f(b)$. Now by substituting y by both a and b and comparing the resulting equations we get

   $$f(a + f(x)) = f(b + f(x)) \quad \forall x \in \mathbb{R}.$$

   By doing the same thing for x we get

   $$a - b = f(a + f(a) + f(y)) - f(b + f(a) + f(y)) \quad \forall y \in \mathbb{R}.$$

   For $a = b$, it is now enough to find $x_0, y_0 \in \mathbb{R}$ such that $f(a) + f(y_0) = f(x_0)$. These numbers can be clearly obtained by putting $x = f(a) - f(y) + f(f(y))$ in the main equation. So f is injective.

   Setting $x = y = 0$, we get $f(2f(0)) = f(0)$ and the injectivity yields $f(0) = 0$. Now setting $x = 0$. we get $f(f(y)) = f(y)$ for all $y \in \mathbb{R}$ and the injectivity implies $f(y) = y$ for all $y \in \mathbb{R}$. This function clearly satisfies the original equation.

4. Let us first try the case when $n = 2$. Put $A = (a_{ij})$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $AB = \begin{pmatrix} a_{12} & 0 \\ a_{22} & 0 \end{pmatrix}$ and $A^2B = \begin{pmatrix} a_{11}a_{12} + a_{12}a_{22} & 0 \\ a_{21}a_{12} + a_{22}^2 & 0 \end{pmatrix}$. Similarly, $BA^2 = \begin{pmatrix} 0 & 0 \\ a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \end{pmatrix}$. On the other hand, $ABA = \begin{pmatrix} a_{12}a_{11} & a_{12}^2 \\ a_{22}a_{11} & a_{22}a_{12} \end{pmatrix}$, so the equality yields $a_{12} = 0$. Then the other corner gives $a_{11}^2 + a_{22}^2 = 2a_{22}a_{11}$ and $a_{11} = a_{22}$. That is, A must be in the form λE for some $\lambda \in \mathbb{R}$, where E is the identity matrix. This form also fits the original equality.

   Trying the same approach in the general case (with $b_{n1} = 1$ and $b_{ij} = 0$ otherwise) would similarly yield that $a_{1n} = 0$. In fact, this also works for any pair of distinct indices k, l.

   Recall that $a_{ij} = e_i A e_j^T$ for all i, j, where e_i is the i-th row of the identity matrix. In particular $E = (\delta_{ij}) = (e_i e_j^T)$.

   For distinct k and l, let $b_{kl} = 1$ and $b_{ij} = 0$ otherwise. It means that $B = e_k^T e_l$, We want to show that $a_{lk} = 0$. For that, let us calculate the element in the same position on both sides of the equation. The first summand of the left hand side gives

   $$e_l A^2 B e_k^T = e_l A^2 e_k^T e_l e_k^T = 0,$$

because $e_l e_k^T = \delta_{lk} = 0$. The second summand is similar and we get 0 on the left hand side. The right hand side gives

$$2e_l ABAe_k^T = 2e_l Ae_k^T e_l Ae_k^T = 2a_{lk}^2.$$

Thus $a_{lk} = 0$. Since k and l were arbitrary, this means that A must be a diagonal matrix.

Let us fix k,l, and B as before and check the kl element of both sides. LHS gives

$$e_k A^2 e_k^T e_l e_l^T + e_k e_k^T e_l A^2 e_l^T = e_k A^2 e_k^T + e_l A^2 e_l^T = (A^2)_{kk} + (A^2)_{ll},$$

because $e_i e_i^T = \delta_{ii} = 1$. But $(A^2)_{ii} = a_{ii}^2$, since A is diagonal. So we get $a_{kk}^2 + a_{ll}^2$.

RHS gives

$$2e_k Ae_k^T e_l Ae_l^T = 2a_{kk}a_{ll}.$$

So $a_{kk}^2 + a_{ll}^2 = 2a_{kk}a_{ll}$, that is $(a_{kk} - a_{ll})^2 = 0$ and $a_{kk} = a_{ll}$. Since k and l were arbitrary we see that A is of the form λE.

5. Let

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

It can be easily seen (e.g., by induction) that $AB = M^k$. Since $|AB| = 1 \neq 0$, also $|A| \neq 0$. Now $BA = A^{-1}ABA = A^{-1}M^kA = (A^{-1}MA)^k$.

6. Let us note that an odd permutation must have an even order. Indeed, let a permutation σ be represented as a product of k transpositions, where k is odd. Then σ^n is represented as a product of kn transpositions. If $\sigma^n = id$, then kn must be even, because id is an even permutation, so n must be even.

Therefore, by (e), at least half of permutations must have an even order.

On the other hand, if $n \geq 4$, then there also are even permutations with an even order, e.g., (12)(34). So in this case, there are more permutations with an even order.

For $n = 2$ and $n = 3$, it is easily checked that the numbers are equal. Even order: (12), (13), (23). Odd order: id, (123), (132).