Tartu Ülikooli üliõpilaste matemaatikaolümpiaad

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1. Tõesta, et paaritu täisarvu n korral
$$\int_0^{2\pi} \sin(\sin x + nx) + \cos(\cos x + nx) dx = 0.$$

2. Olgu P ja Q erinevad reaalmuutuja polünoomid ning kehtigu polünoomide võrdus $P(Q(x)) = Q(P(x))$ iga $x \in \mathbb{R}$ korral. Tõesta, et polünoom $P(P(x)) - Q(Q(x))$ jagub polünoomiga $P(x) - Q(x)$.

3. Olgu $f \in C^\infty(\mathbb{R})$ (lõpmata palju kordi diferentseeruv funktsioon), kusjuures $f(0) = 0$, $f^{(k)}(0) = 0$ ja $f^{(k)}(x) \geq 0$ kõikide $k \in \mathbb{N}$ ja $x > 0$ korral. Tõesta, et $f(x) = 0$ iga $x > 0$ korral.

4. Jada (x_n) on defineeritud seostega $x_1 = a > 1$ ja $x_{n+1} = x_n^2 - x_n + 1$, $n \geq 1$. Leia
$$\sum_{n=1}^{\infty} \frac{1}{x_n}.$$

5. Olgu funktsioon f pidevalt diferentseeruv vahemikus $(0, \infty)$.
   (a) On teada, et $f(x) + f'(x) \to A$ protsessis $x \to \infty$. Tõesta, et $f(x) \to A$ protsessis $x \to \infty$.
   (b) On teada, et $f(x) - f'(x) \to A$ protsessis $x \to \infty$. Kas võib väita, et $f(x) \to A$ protsessis $x \to \infty$?

6. Olgu $a_1 < a_2 < \ldots < a_n$ ja $b_1 < b_2 < \ldots < b_n$ reaalarvud. Tõesta, et
$$\begin{vmatrix} e^{a_1 b_1} & e^{a_1 b_2} & \ldots & e^{a_1 b_n} \\ e^{a_2 b_1} & e^{a_2 b_2} & \ldots & e^{a_2 b_n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{a_n b_1} & e^{a_n b_2} & \ldots & e^{a_n b_n} \end{vmatrix} > 0.$$

1. Let n be an odd integer. Prove the equality

$$\int_0^{2\pi} \sin(\sin x + nx) + \cos(\cos x + nx) dx = 0.$$

2. Let P and Q be different polynomials with real coefficients such that $P(Q(x)) = Q(P(x))$ for all $x \in \mathbb{R}$. Prove that the polynomial $P(P(x)) - Q(Q(x))$ is divisible by the polynomial $P(x) - Q(x)$.

3. Let $f \in C^\infty(\mathbb{R})$ be a smooth function such that $f(0) = 0$, $f^{(k)}(0) = 0$, and $f^{(k)}(x) \geq 0$ for all $k \in \mathbb{N}$ and $x > 0$. Prove that$f(x) = 0$ for all $x > 0$.

4. The sequence (x_n) is defined by the relations $x_1 = a > 1$ and $x_{n+1} = x_n^2 - x_n + 1$, $n \geq 1$. Find

$$\sum_{n=1}^{\infty} \frac{1}{x_n}.$$

5. Let the function f be continuously differentiable on the interval $(0, \infty)$.
   (a) Assume $f(x) + f'(x) \to A$ for $x \to \infty$. Prove that $f(x) \to A$ for $x \to \infty$.
   (b) Assume $f(x) - f'(x) \to A$ for $x \to \infty$. Is it possible to claim $f(x) \to A$ for $x \to \infty$?

6. Let $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_n$ be real numbers. Prove that

$$\begin{vmatrix} e^{a_1 b_1} & e^{a_1 b_2} & \ldots & e^{a_1 b_n} \\ e^{a_2 b_1} & e^{a_2 b_2} & \ldots & e^{a_2 b_n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{a_n b_1} & e^{a_n b_2} & \ldots & e^{a_n b_n} \end{vmatrix} > 0.$$

Solutions

1. *(∼ Kiev NU)* Substitute $x = y+\pi$ and note that $\sin(y+\pi) = -\sin y$, $\cos(y+\pi) = -\cos y$, $\sin(ny+n\pi) = (-1)^n \sin(ny)$, $\cos(ny + n\pi) = (-1)^n \cos(ny)$. We have

$$\begin{aligned} A &:= \int_0^{2\pi} \sin(\sin x + nx) + \cos(\cos x + nx)\, dx \\ &= \int_0^{2\pi} \sin(\sin x)\cos(nx) + \cos(\sin x)\sin(nx) \\ &\quad + \cos(\cos x)\cos(nx) - \sin(\cos x)\sin(nx)\, dx \\ &= \int_{-\pi}^{\pi} \sin(-\sin y)(-1)^n \cos(ny) + \cos(-\sin y)(-1)^n \sin(ny) \\ &\quad + \cos(-\cos y)(-1)^n \cos(ny) - \sin(-\cos y)(-1)^n \sin(ny)\, dy \end{aligned}$$

In the latter integral three summands are odd functions of y (hence their integral with symmetric bounds is 0) and one is even. So

$$A = (-1)^n \int_{-\pi}^{\pi} \cos(\cos y)\cos(ny)\, dy = 2 \cdot (-1)^n \int_0^{\pi} \cos(\cos y)\cos(ny)\, dy.$$

Again, substitute $y = z+\pi/2$ and note that $\cos(z+\pi/2) = -\sin z$ and $\cos(nz+n\pi/2) = (-1)^{\frac{n+1}{2}} \sin(nz)$. We have

$$A = 2 \cdot (-1)^n \int_{-\pi/2}^{\pi/2} \cos(-\sin z)(-1)^{\frac{n+1}{2}} \sin(nz)\, dz = 0$$

as it is an integral with symmetric bounds of an odd function.

2. *(MSU MechMath)* Note that if P, Q, R are polynomials, then $P(Q(x)) - P(R(x))$ is divisible by $Q(x) - R(x)$ because so is $Q(x)^n - R(x)^n$ for all $n \in \mathbb{N}$. Hence,

$$P(P(x)) - Q(Q(x)) = (P(P(x)) - P(Q(x))) + (Q(P(x)) - Q(Q(x)))$$

is divisible by $P(x) - Q(x)$.

3. *(Yugoslavia)* Take $x > 0$. Taylor expansion at 0 gives

$$f(x) = \frac{f^{(k)}(\theta)}{k!} x^k, \quad 0 \le \theta \le x.$$

Since $f^{(k)}$ is non-decreasing on $[0, \infty)$, we have

$$\frac{f^{(k)}(x)}{k!} x^k \ge f(x).$$

Taylor expansion at x gives

$$f(2x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} x^k + \frac{f^{(n)}(\theta)}{n!} x^n \ge \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} x^k \ge n f(x)$$

for some $\theta \in [x, 2x]$. The latter inequality can hold for all n only if $f(x) = 0$.

4. *(Math.Battle)*

Answer: $\dfrac{1}{a-1}$.

Clearly, (x_n) is non-decreasing. Assume that $c = \lim x_n < \infty$. Then $c^2 - 2c + 1 = 0$, so that $c = 1$, which contradicts $a > 1$. Therefore, $x_n \to \infty$. Now

$$\frac{1}{x_1 - 1} = \frac{1}{x_1} + \frac{1}{x_1^2 - x_1} = \frac{1}{x_1} + \frac{1}{x_2 - 1} = \ldots = \sum_{n=1}^{N} \frac{1}{x_n} + \frac{1}{x_{N+1} - 1},$$

so that

$$\frac{1}{a-1} - \sum_{n=1}^{N} \frac{1}{x_n} = \frac{1}{x_{N+1} - 1} \xrightarrow[N\to\infty]{} 0.$$

5. *(USSR tour)*
   (a) Fix $\varepsilon > 0$. There exists x_0 such that $|f(x) + f'(x) - A| < \varepsilon$, i.e.,

   $$-f(x) + A - \varepsilon < f'(x) < -f(x) + A + \varepsilon$$

   for all $x \geq x_0$.
   Let f_1 and f_2 be solutions to differential equations $f'(x) = -f(x) + A - \varepsilon$ and $f'(x) = -f(x) + A + \varepsilon$, respectively, such that $f(x_0) = f_1(x_0) = f_2(x_0)$. Then $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \geq x_0$. Indeed, put $g(x) = f(x) - f_1(x)$. Then $g(x_0) = 0$ and $g'(x) \geq -g(x)$ for all $x \geq x_0$. Assume that $g(x) < 0$ for some $x > x_0$. Let g attain its minimum value in $[0, x]$ at $\theta \in (0, x]$. Then $g(\theta) < 0$ and $g'(\theta) \geq -g(\theta) > 0$, so that $g(\theta - \delta) < g(\theta)$ for some small $\delta > 0$, a contradiction. That is, $f(x) \geq f_1(x)$ for all $x \geq x_0$. Similarly, $f(x) \leq f_2(x)$ for all $x \geq x_0$.
   By solving the differential equations we obtain

   $$c_1 e^{-x} + A - \varepsilon \leq f(x) \leq c_2 e^{-x} + A + \varepsilon$$

   for some constants c_1, c_2 and all $x \geq x_0$. Hence, there exists x_1 such that

   $$A - 2\varepsilon \leq f(x) \leq A + 2\varepsilon$$

   for all $x \geq x_1$.
   (b) No. Take any $A \in \mathbb{R}$ and put $f(x) = e^x + A$. Then $f(x) - f'(x) = A$ for all $x \in \mathbb{R}$ but $f(x) \to \infty$ when $x \to \infty$.

6. *(Géza Kós, Budapest)* Denote the determinant by D. Apply induction on n. For $n = 1$ the statement is obvious. Suppose $n > 1$ and assume the statement is true for all smaller values.
   Let $c_i = a_i - a_1$. Then

   $$D = \begin{vmatrix} e^{a_1 b_1} & e^{a_1 b_2} & \ldots & e^{a_1 b_n} \\ e^{a_1 b_1} e^{c_2 b_1} & e^{a_1 b_2} e^{c_2 b_2} & \ldots & e^{a_1 b_n} e^{c_2 b_n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{a_1 b_1} e^{c_n b_1} & e^{a_1 b_2} e^{c_n b_2} & \ldots & e^{a_1 b_n} e^{c_n b_n} \end{vmatrix} = e^{a_1(b_1 + b_2 + \ldots + b_n)} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ e^{c_2 b_1} & e^{c_2 b_2} & \ldots & e^{c_2 b_n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_n b_1} & e^{c_n b_2} & \ldots & e^{c_n b_n} \end{vmatrix},$$

   and it is sufficient to prove that the last determinant D_1 is positive.
   Eliminate the first row by multiplying with

   $$\begin{pmatrix} 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix},$$

so that

$$D_1 = \begin{vmatrix} 1 & 0 & 0 & \dots & 1 \\ e^{c_2 b_1} & e^{c_2 b_2} - e^{c_2 b_1} & e^{c_2 b_3} - e^{c_2 b_2} & \dots & e^{c_2 b_n} - e^{c_2 b_{n-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{c_n b_1} & e^{c_n b_2} - e^{c_n b_1} & e^{c_n b_3} - e^{c_n b_2} & \dots & e^{c_n b_n} - e^{c_n b_{n-1}} \end{vmatrix}$$
$$= \begin{vmatrix} e^{c_2 b_2} - e^{c_2 b_1} & e^{c_2 b_3} - e^{c_2 b_2} & \dots & e^{c_2 b_n} - e^{c_2 b_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_n b_2} - e^{c_n b_1} & e^{c_n b_3} - e^{c_n b_2} & \dots & e^{c_n b_n} - e^{c_n b_{n-1}} \end{vmatrix}.$$

Consider the function

$$f(t) = \begin{vmatrix} e^{c_2 t} & e^{c_2 b_3} - e^{c_2 b_2} & \dots & e^{c_2 b_n} - e^{c_2 b_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_n t} & e^{c_n b_3} - e^{c_n b_2} & \dots & e^{c_n b_n} - e^{c_n b_{n-1}} \end{vmatrix}.$$

By Lagrange's mean value theorem there exists $\xi_1 \in (b_1, b_2)$ such that $f(b_1) - f(b_2) = (b_2 - b_1) f'(t)$, i.e.,

$$\begin{vmatrix} e^{c_2 b_2} - e^{c_2 b_1} & e^{c_2 b_3} - e^{c_2 b_2} & \dots & e^{c_2 b_n} - e^{c_2 b_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_n b_2} - e^{c_n b_1} & e^{c_n b_3} - e^{c_n b_2} & \dots & e^{c_n b_n} - e^{c_n b_{n-1}} \end{vmatrix} = (b_2 - b_1) \begin{vmatrix} c_2 e^{c_2 \xi_1} & e^{c_2 b_3} - e^{c_2 b_2} & \dots & e^{c_2 b_n} - e^{c_2 b_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ c_n e^{c_n \xi_1} & e^{c_n b_3} - e^{c_n b_2} & \dots & e^{c_n b_n} - e^{c_n b_{n-1}} \end{vmatrix}.$$

Repeating the same argument for each column, we can find real numbers $\xi_i \in (b_i, b_{i+1})$, $1 \le i \le n-1$, such that

$$\begin{vmatrix} e^{c_2 b_2} - e^{c_2 b_1} & e^{c_2 b_3} - e^{c_2 b_2} & \dots & e^{c_2 b_n} - e^{c_2 b_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_n b_2} - e^{c_n b_1} & e^{c_n b_3} - e^{c_n b_2} & \dots & e^{c_n b_n} - e^{c_n b_{n-1}} \end{vmatrix} = \left(\prod_{i=1}^{n-1} (b_{i+1} - b_i) \right) \begin{vmatrix} c_2 e^{c_2 \xi_1} & c_2 e^{c_2 \xi_2} & \dots & c_2 e^{c_2 \xi_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ c_n e^{c_n \xi_1} & c_n e^{c_n \xi_2} & \dots & c_n e^{c_n \xi_{n-1}} \end{vmatrix}$$
$$= \left(\prod_{i=1}^{n-1} c_{i+1} (b_{i+1} - b_i) \right) \begin{vmatrix} e^{c_2 \xi_1} & e^{c_2 \xi_2} & \dots & e^{c_2 \xi_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{c_n \xi_1} & e^{c_n \xi_2} & \dots & e^{c_n \xi_{n-1}} \end{vmatrix}.$$

The latter determinant is positive by the induction hypothesis.