Tartu Ülikooli üliõpilaste matemaatikaolümpiaad

1. Funktsioon $f \in C^2[0,1]$ (kaks korda pidevalt diferentseeruv lõigus $[0,1]$) rahuldab tingimusi:

   (a) $f(0) = f'(0) = 1$,

   (b) $f''(x) \geq 0$ iga $x \in (0,1)$ korral,

   (c) $\int_0^1 f(x)dx = \frac{3}{2}$.

   Leia kõik sellised funktsioonid f.

2. Olgu $x_1, x_2, \ldots, x_n$ nullist erinevad elemendid vektorruumis ja olgu A selline lineaarne teisendus, et

   $$Ax_1 = x_1$$

   ja

   $$Ax_k = x_k + x_{k-1},\ k = 2, 3, \ldots, n.$$

   Tõesta, et elemendid $x_1, \ldots, x_n$ on lineaarselt sõltumatud.

3. Olgu $P(x) = ax^2 + bx + c$, kus $a, b, c \in \mathbb{R}$. Seejuures $0 \leq P(-1) \leq 1$, $0 \leq P(0) \leq 1$ ja $0 \leq P(1) \leq 1$. Tõesta, et $P(x) \leq 9/8$ iga $x \in [0,1]$ korral.

4. Tasandil on antud n punkti. Tõesta, et kehtib üks järgmistest võimalustest:

   - kas kõik need punktid asetsevad ühel ja samal sirgel,
   - või leidub sirge, mis läbib täpselt 2 punkti.

5. Olgu antud 2013×2013 ruutmaatriksid A ja B, kusjuures $AB = 0$. Tõesta, et vähemalt üks maatriksidest $A + A^T$ ja $B + B^T$ on singulaarne.

6. Tähistame

   $$M := \{x \in \mathbb{R} : \text{rida} \sum_{n=0}^{\infty} \sin(n!\pi x) \text{ koondub}\}.$$

   Tõesta, et

   (a) M on kõikjal tihe, s.t. suvalise lõigu ja hulga M ühisosa ei ole tühi, (5 punkti)

   (b) $e \in M$, (15 punkti)

   (c) hulga M sees ei leidu ühtegi lõiku. (20 punkti)

1. Find all functions $f \in C^2[0,1]$ (two times continuously differentiable on the interval $[0,1]$) such that:
   (a) $f(0) = f'(0) = 1$,
   (b) $f''(x) \geq 0$ for all $x \in (0,1)$,
   (c) $\int_0^1 f(x)dx = \frac{3}{2}$.

2. Let $x_1, x_2, \ldots, x_n$ be nonzero elements in a vector space and let A be a linear transformation such that

$$Ax_1 = x_1$$

and

$$Ax_k = x_k + x_{k-1}, \ k = 2, 3, \ldots, n.$$

   Prove that elements $x_1, \ldots, x_n$ are linearly independent.

3. Let $P(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$, satisfy $0 \leq P(-1) \leq 1$, $0 \leq P(0) \leq 1$, and $0 \leq P(1) \leq 1$. Prove that $P(x) \leq 9/8$ for all $x \in [0,1]$.

4. Take n points on a plane. Prove that one of the following conditions holds:
   - all of these points lie on the same line,
   - there exists a line with exactly two of these points.

5. Let A and B be 2013×2013 square matrices such that $AB = 0$. Prove that at least one of the matrices $A + A^T$ and $B + B^T$ is not invertible.

6. Denote

$$M := \{x \in \mathbb{R} : \text{the series} \sum_{n=0}^{\infty} \sin(n!\pi x) \text{ converges}\}.$$

   Prove that
   (a) M is dense, i.e., the intersection of any interval and M is not empty, (5 points)
   (b) $e \in M$, (15 points)
   (c) M does not contain any interval. (20 points)

2 Solutions

1. By (b) we know that $f'(x) \geq f'(0) = 1$ for all $x \in [0, 1]$. Then $f(x) - f(0) = \int_0^x f'(y)dy \geq \int_0^x dy = x$, so that $f(x) \geq 1 + x$. Now $f(x) - 1 - x$ is a continuous function on $[0, 1]$ such that
$$\int_0^1 |f(x) - 1 - x|dx = \int_0^1 f(x)dx - \int_0^1 (1 + x)dx = \frac{3}{2} - \frac{3}{2} = 0.$$
Hence, $|f(x) - 1 - x| = 0$ for all $x \in [0, 1]$. That is, $f(x) = x + 1$.

2. Let us prove it by induction. The case $n = 1$ is obvious. Assume the claim is true for $n - 1$. If $x_1, \ldots, x_n$ were linearly dependent, then $c_1x_1 + \cdots + c_nx_n = 0$ for some non-zero vector $(c_1, \ldots, c_n)$. Then $c_1Ax_1 + \cdots + c_nAx_n = 0$. Substracting the former equation from the latter, and noting that $Ax_1 - x_1 = 0$ and $Ax_k - x_k = x_{k-1}$ for $k \geq 1$, we get $c_2x_1 + \cdots + c_nx_{n-1} = 0$. By assumption, $c_2 = c_3 = \cdots = c_n = 0$, so that $c_1x_1 = 0$. But $x_1 \neq 0$ and $c_1 = 0$ contradicts the choice of c_1.

3. Put $Q(x) := P(x) - 1/2$. Then $|Q(-1)|, |Q(0)|, |Q(1)| \leq 1/2$. The Lagrange polynomial for $\{-1, 0, 1\}$ must coincide with $Q(x)$, so
$$Q(x) = \frac{x(x-1)}{2}Q(-1) + (1 - x^2)Q(0) + \frac{x(x+1)}{2}Q(1).$$
Now for $x \in [-1, 1]$ we have
$$Q(x) \leq \frac{1}{2} \max_{x \in [-1,1]} \left(\left| \frac{x(x-1)}{2} \right| + \left| 1 - x^2 \right| + \left| \frac{x(x+1)}{2} \right| \right) = \frac{1}{2} \cdot \frac{5}{4} = \frac{5}{8}.$$
Then $P(x) = \frac{1}{2} + Q(x) \leq \frac{9}{8}$ for all $x \in [-1, 1]$.

4. The case $n \leq 2$ is obvious. Let $n \geq 3$ and assume that for every two points there is a third one on the same line. Choose a pair consisting of a line defined by two points A and B (which must also have a third point C) and of a point D outside this line such that the distance from D to the line is the least possible (among all the pairs). Let H be a point on this line (which is not necessarily one of the n points) such that $DH \perp AB$. Of A, B, C at least two points are on the same side with respect to H. Without loss of generality we can assume that B is between A and H. Now the distance from B to the line AD is less than $|DH|$, which contradicts the choice of our pair. Therefore, there is no such pair and all the points lie on the same line.

5. **Lemma 1.** If A and B are matrices of the same size, then $\operatorname{rank}(A + B) \leq \operatorname{rank} A + \operatorname{rank} B$.

   Lemma 2. If A and B are square matrices of order n, then $\operatorname{rank} A + \operatorname{rank} B \leq \operatorname{rank}(AB) + n$.

   By Lemma 2,
$$\operatorname{rank} A + \operatorname{rank} B \leq \operatorname{rank}(AB) + 2013 = 2013.$$
Hence, one of the numbers $\operatorname{rank} A$, $\operatorname{rank} B$ is not greater than $\lfloor 2013/2 \rfloor = 1006$. Let us assume that $\operatorname{rank} A \leq 1006$, then $\operatorname{rank} A^T = \operatorname{rank} A \leq 1006$. By Lemma 1,
$$\operatorname{rank}(A + A^T) \leq \operatorname{rank} A + \operatorname{rank} A^T \leq 2012 < 2013.$$

6. (a) In any interval there is a rational point p/q, $q > 0$. If $n \geq q$ then $n!x = p(q-1)(q+1)\ldots n$ is an integer, so that $\sin(n!\pi x) = 0$. Hence, the series
$$\sum_{n=0}^{\infty} \sin(n!\pi x)$$
converges.

(b) The Taylor expansion of e^t at $t = 1$ yields

$$n!e = \sum_{k=0}^{\infty} \frac{n!}{k!} = \sum_{k=0}^{n-2} \frac{n!}{k!} + (n+1) + \frac{1}{n+1} + \sum_{k=n+2}^{\infty} \frac{n!}{k!}$$

for $n \geq 2$. Note that

$$\sum_{k=0}^{n-2} \frac{n!}{k!} = n(n-1) \sum_{k=0}^{n-2} \frac{(n-2)!}{k!}$$

is an even number and

$$\sum_{k=n+2}^{\infty} \frac{n!}{k!} = \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \leq \sum_{m=0}^{\infty} \frac{1}{(n+1)(n+2)(n+3)^m}$$

$$= \frac{n+3}{(n+1)(n+2)^2} = O\left(\frac{1}{n^2}\right).$$

Note that $|\sin x - \sin y| \leq |x - y|$, so that $\sin(x + y) = \sin x + z$, where $|z| \leq |y|$. Therefore,

$$\sin(n!\pi e) = \sin\left(\pi \sum_{k=0}^{n-2} \frac{n!}{k!} + \pi(n+1) + \frac{\pi}{n+1} + \pi \sum_{k=n+2}^{\infty} \frac{n!}{k!}\right) = \sin\left(\pi(n+1) + \frac{\pi}{n+1} + \pi \sum_{k=n+2}^{\infty} \frac{n!}{k!}\right)$$

$$= \sin\left(\pi(n+1) + \frac{\pi}{n+1}\right) + \theta_n = (-1)^{n+1} \sin(\frac{\pi}{n+1}) + \theta_n,$$

where $|\theta_n| \leq \pi \sum_{k=n+2}^{\infty} \frac{n!}{k!} = O\left(\frac{1}{n^2}\right)$. Now the series

$$\sum_{n=2}^{\infty} (-1)^{n+1} \sin(\frac{\pi}{n+1})$$

converges by the alternating series test and the series

$$\sum_{n=2}^{\infty} \theta_n$$

converges absolutely. Hence, the series

$$\sum_{n=0}^{\infty} \sin(n!\pi e) = 2 \sin \pi e + \sum_{n=2}^{\infty} (-1)^{n+1} \sin(\frac{\pi}{n+1}) + \sum_{n=2}^{\infty} \theta_n$$

converges as well.

(c) Let us prove that any interval $[a, b]$, $a < b$, contains a point $x \notin M$. Take a natural number n_1 such that $2/n_1! < b - a$. Then there exists an integer m_1 such that

$$a \leq \frac{m_1}{n_1!} < \frac{(m_1 + 1)}{n_1!} \leq b.$$

Put

$$a_1 := \frac{(m_1 + \frac{1}{3})}{n_1!}, \quad b_1 := \frac{(m_1 + \frac{2}{3})}{n_1!}.$$

Note that $[a_1, b_1] \subset [a, b]$ and for any $x \in [a_1, b_1]$ one has

$$|\sin n_1! \pi x| \geq \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Let us now take $n_2 > n_1$ and m_2 such that $2/n_2! < b_1 - a_1$ and

$$a_1 \leq \frac{m_2}{n_2!} < \frac{(m_2 + 1)}{n_2!} \leq b_1.$$

Denoting

$$a_2 := \frac{(m_2 + \frac{1}{3})}{n_2!}, \quad b_2 := \frac{(m_2 + \frac{2}{3})}{n_2!}.$$

we get $[a_2, b_2] \subset [a_1, b_1]$ and $|\sin n_2! \pi x| \geq \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ for any $x \in [a_2, b_2]$.
If we continue in this way, we obtain a system of nested intervals

$$[a, b] \supset [a_1, b_1] \supset [a_2, b_2] \supset \dots$$

and an increasing sequence of natural numbers $n_1 < n_2 < \dots$ such that $|\sin n_k! \pi x| \geq \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ for all $x \in [a_k, b_k]$.
Taking $x \in \bigcap_{k=1}^{\infty} [a_k, b_k]$ we get $|\sin n_k! \pi x| \geq \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ for $k = 1, 2, \dots$. Thus the series

$$\sum_{n=0}^{\infty} \sin(n! \pi x)$$

diverges and $x \notin M$.