Tartu Ülikooli üliõpilaste matemaatikaolümpiaad

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1. Olgu $f : \mathbb{R} \to \mathbb{R}$ integreeruv ja $0 < f(z) < \frac{1}{2z}$ iga $z > 0$ korral. Tõesta, et funktsioon

$$g(x) := \int_0^x z^2 f(z)dz - \left(\int_0^x z f(z)dz\right)^2$$

on kasvav vahemikus $(0, \infty)$.

2. Iga arvrida, mille elemendid on paarikaupa erinevad hulga $S \subset \mathbb{R}$ elemendid, on koonduv. Tõesta, et S on ülimalt loenduv.

3. Olgu $n \in \mathbb{N}$. Maatriksi $A \in \mathrm{Mat}_n(\mathbb{R})$ peadiagonaalil on ainult nullid ning kõik teised elemendid on kas 1 või 2018. Tõesta, et maatriksi A astak on kas n või $n-1$.

4. Vaatleme vektorruume $\mathbb{R}^2$, $\mathbb{R}^3$ ja c_{00} üle $\mathbb{R}$, kus

$$c_{00} = \{x = (x_n) : \mathbb{N} \to \mathbb{R} \mid \exists N \in \mathbb{N}\ \forall n \geq N\ x_n = 0\}.$$

Ruumides $\mathbb{R}^2$, $\mathbb{R}^3$ olgu normiks harilik eukleidiline norm:

$$\|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}, \qquad \|(x_1, x_2, x_3)\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Ruum c_{00} olgu varustatud järgmise normiga:

$$\|x\| = \left(\sum_{n\in\mathbb{N}} |x_n|^2\right)^{1/2}.$$

Iga nimetatud ruumi korral leia vähim võimalik kinniste kerade arv, mis katavad terve ühiksfääri $\{x\colon \|x\| = 1\}$, kuid ükski kera ei sisalda nullpunkti, või tõesta, et lõplikust arvust keradest ei piisa. See tähendab, leia vähim $m \in \mathbb{N}$, mille korral leiduvad elemendid $x^1, \ldots, x^m$ ja raadiused $r_i \in (0, \|x^m\|)$, $i = 1, \ldots, m$ nii, et iga elemendi y, $\|y\| = 1$, korral leidub indeks $i \in \{1, \ldots, m\}$ omadusega $\|y - x^i\| \leq r_i$.

5. Olgu $n \in \mathbb{N}$ ja $c > 0$. Polünoomi P kõik n nullkohta on erinevad reaalarvud. Tõesta, et hulk

$$\{x \in \mathbb{R}\colon P(x) \neq 0,\ \frac{P'(x)}{P(x)} > c\}$$

on lõpliku arvu vahemike ühend, kusjuures nende vahemike pikkuste summa võrdub $\frac{n}{c}$.

6. Olgu n-tipulise ($n \in \mathbb{N}$) täisgraafi K_n iga serv värvitud ühte kolmest värvist. Tõesta, et sellel graafil leidub sidus alamgraaf, mille kõik servad on sama värvi ja mille tippude arv on vähemalt $n/2$.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable and such that $0 < f(z) < \frac{1}{2z}$ for all $z > 0$. Prove that the function

$$g(x) := \int_0^x z^2 f(z)dz - \left(\int_0^x zf(z)dz\right)^2$$

is increasing on $(0, \infty)$.

2. Every series composed of distinct elements from a set $S \subset \mathbb{R}$ is convergent. Prove that S is at most countable.

3. Let $n \in \mathbb{N}$. Consider matrix A of size $n \times n$ such that there are only zeroes on the main diagonal and all the other elements are either 1 or 2018. Prove that the rank of A is either n or $n-1$.

4. Consider the vector spaces $\mathbb{R}^2$, $\mathbb{R}^3$, and

$$c_{00} = \{x = (x_n) : \mathbb{N} \to \mathbb{R} \mid \exists N \in \mathbb{N}\ \forall n \geq N\ x_n = 0\}$$

Let the spaces $\mathbb{R}^2$, $\mathbb{R}^3$ be equipped with the usual Euclidean distance:

$$\|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}, \qquad \|(x_1, x_2, x_3)\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Let c_{00} be equipped with the norm

$$\|x\| = \left(\sum_{n\in\mathbb{N}} |x_n|^2\right)^{1/2}.$$

For all these spaces find the minimal finite number of closed balls not containing the origin such that their union contains the unit sphere $\{x\colon \|x\| = 1\}$ or prove that it takes infinite number of balls to cover the unit sphere in this way. That is, find a minimal $m \in \mathbb{N}$ for which there exist elements $x^1, \ldots, x^m$ and radii $r_i \in (0, \|x^m\|)$, $i = 1, \ldots, m$, such that for every element y satisfying $\|y\| = 1$ there exists an index $i \in \{1, \ldots, m\}$ with $\|y - x^i\| \leq r_i$.

5. Let $n \in \mathbb{N}$ and let $c > 0$. All the n roots of a polynomial P are distinct real numbers. Prove that the set

$$\{x \in \mathbb{R}\colon P(x) \neq 0,\ \frac{P'(x)}{P(x)} > c\}$$

is a union of a finite number of intervals such that the sum of lengths of these intervals equals $\frac{n}{c}$.

6. Let $n \in \mathbb{N}$. In a complete graph with n vertices K_n every edge is colored with one of three colors. Prove that in this graph there exists a connected subgraph with edges of the same color and at least $n/2$ vertices.

Solutions

1. Note that
$$g'(x) = x^2 f(x) - 2xf(x) \int_0^x zf(z)dz > x^2 f(x) - xf(x) \int_0^x dz = 0.$$

2. Consider sets
$$S_n^+ = \{x \in S \mid x > \frac{1}{n}\}$$
and
$$S_n^- = \{x \in S \mid x < -\frac{1}{n}\}.$$
If any of them is infinite we can construct a divergent series from distinct elements of it. So each of them is finite and
$$\{0\} \cup \bigcup_{n \in \mathbb{N}} (S_n^+ \cup S_n^-)$$
is countable. But the latter set clearly contains S.

3. Let $B = (b_{ij})$ with $b_{ij} = 1$ for all i, j. Note that $A - B$ modulo 2017 is a diagonal matrix with -1 on the diagonal. So $|A - B|$ is not 0 modulo 2017 and $\operatorname{rank}(A - B) = n$. Note that rank of a linear transformation L on $\mathbb{R}^n$ is just the dimension of the image $L(\mathbb{R}^n)$. Since $(A-B)(\mathbb{R}^n) \subset A(\mathbb{R}^n)+B(\mathbb{R}^n)$, we have $n = \operatorname{rank}(A - B) \leq \operatorname{rank} A + \operatorname{rank} B = \operatorname{rank} A + 1$.

4. In $\mathbb{R}^n$ the minimal number is $n + 1$. We cannot cover with n balls because any ball leaves a diametral section of the ball intact and by induction you cannot cover it by $n - 1$. The base $(n = 1)$ is clear. On the other hand, note that for any half-space $H = H(y, \alpha) = \{x \in \mathbb{R}^n \mid \langle x, y\rangle \geq \alpha\}$ defined by $y \in \mathbb{R}^n$ and $\alpha > 0$ not containing the origin you can find a large enough ball B not containing the origin such that $S \cap H \subset S \cap B$ (S denotes the sphere). Now you can clearly cover the sphere by halfspaces $H(e^i, \alpha_i)$, $i = 1, \ldots, n$ and $H(-(e^1 + \cdots + e^n), \alpha)$ for small enough α_i and α (an element e^i is defined by $e^i_k = \delta_{ik}$, i.e., 1 when $i = k$ and 0 otherwise).

   In c_{00}, you cannot cover by a finite number of balls. Take $x^1, \ldots, x^m$. We can find $n \in \mathbb{N}$ such that $x^i_n = 0$ for $i = 1, \ldots, m$. Then for any i,
$$\|x^i - e^n\| = \sqrt{\|x^i\|^2 + 1} > \|x^i\|.$$
   So $e^n \in S$ is not in any ball with center x^i that does not contain the origin.

5. Take the roots $\{x_1, \ldots, x_n\}$ in increasing order. The function
$$f(x) = \frac{P'(x)}{P(x)} = \frac{1}{x - x_1} + \cdots + \frac{1}{x - x_n}$$
   is decreasing on intervals $(-\infty, x_1)$, $(x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, \infty)$ from ∞ to $-\infty$ or 0, because
$$f'(x) = -\left(\frac{1}{(x - x_1)^2} + \cdots + \frac{1}{(x - x_n)^2}\right) < 0.$$
   Note that $f(x) < 0$ on $(-\infty, x_1)$. The set in question is a union of n intervals $(x_1, y_1), \ldots, (x_n, y_n)$ with $y_k \in (x_k, x_{k+1})$ for $k = 1, \ldots, n - 1$, $y_n \in (x_n, \infty)$, and $f(y_1) = \cdots = f(y_n) = c$.

   Note that $y_1, \ldots, y_n$ are n distinct roots of the polynomial $cP - P'$. Let $P = a_n x^n + \ldots a_1 x + a_0$. A Vieta's formula now gives
$$x_1 + \cdots + x_n = -\frac{a_{n-1}}{a_n},$$
$$y_1 + \cdots + y_n = \frac{na_n - ca_{n-1}}{ca_n} = \frac{n}{c} - \frac{a_{n-1}}{a_n}.$$

6. Let A be the set of vertices of the original graph. Find the largest connected single-coloured subgraph G_1 with its set of vertices A_1 and assume that $|A_1| < n/2$. Take any $x \in A_1$ and $y \notin A_1$. The edge xy is not of the same colour as G_1 by the maximality of G_1. Let G_2 denote the largest connected single-coloured subgraph containing xy and let A_2 be its set of vertices. Note that

   - $x \in A_1 \cap A_2 \neq \emptyset$,
   - $y \in A_2 \setminus A_1 \neq \emptyset$,
   - $A_1 \setminus A_2 \neq \emptyset$, because otherwise $|A_1| < |A_2|$ contradicts the maximality of G_1,
   - $A \setminus (A_1 \cup A_2) \neq \emptyset$, because $|A_1 \cup A_2| < 2 \cdot n/2 = n$.

   Any edge between $A_2 \setminus A_1$ and $A_1 \setminus A_2$ must be of the third color (by the maximality of G_1 and G_2). These edges form a connected subgraph with the set of vertices $A_1 \triangle A_2 := (A_2 \setminus A_1) \cup (A_1 \setminus A_2)$. The same is true for edges between $A_1 \cap A_2$ and $A \setminus (A_1 \cup A_2)$. These form a connected subgraph with the set of vertices $A \setminus (A_1 \triangle A_2)$. One of $A \setminus (A_1 \triangle A_2)$ and $A_1 \triangle A_2$ must have at least $n/2$ vertices.