Matemaatika võistlus

Tartu, 08.03.2019

1. Olgu p algarv. On teada, et ükski täisarvudest $a_1, \ldots, a_{p-1}$ ei jagu arvuga p. Kas alati iga $r = 1, \ldots, p-1$ jaoks leidub selline indeksite hulk $I \subseteq \{1, \ldots, p-1\}$ nii, et summa $\sum_{i \in I} a_i$ jagamisel arvuga p annab jäägi r?

2. Olgu funktsioon $f : [0,1] \to [0,1]$ pidev ja mittekahanev lõigus $[0,1]$ ja diferentseeruv vahemikus $(0,1)$, kusjuures $f'(x) < 1$ iga $x \in (0,1)$ korral. Tähistame $A_0 = [0,1]$ ja $A_{n+1} = f(A_n)$ iga $n = 0, 1, \ldots$ jaoks. Tõesta, et $\sup_{x,y \in A_n} |x - y| \to 0$.

3. Olgu n ja m positiivsed täisarvud ning olgu $A_1, \ldots, A_m$ idempotentsed $(n \times n)$-maatriksid (st $A^2 = A$). Tähistagu I samasusteisendust. Tõesta, et

$$\sum_{i=1}^{m} \dim \ker A_i \geq \operatorname{rank}(I - A_1 A_2 \ldots A_m).$$

   (Siin dim tähendab vektorruumi mõõdet; $\ker A = A^{-1}(0)$ on maatriksi nullruum; rank on maatriksi astak.)

4. Ühe keele tähestik koosneb n tähest, kusjuures kõik sõnad on lõplikud ning ükski sõna ei ole mingi teise sõna alguseks. Tähistagu a_k selle keele nende sõnade arvu, mille pikkus on $k \in \mathbb{N}$. Tõesta, et

$$\sum_{k \geq 1} \frac{a_k}{n^k} \leq 1.$$

5. Olgu $x > 0$. Tõesta, et

$$\lim_{n \to \infty} \frac{1^{x-1} + 2^{x-1} + \cdots + n^{x-1}}{n^x} = \frac{1}{x}.$$

Math competition

Tartu, 08.03.2019

1. Let p be a prime number, which does not divide any of the integers $a_1, \dots, a_{p-1}$. Is it always true, that for any $r = 1, \dots, p-1$ one can find a subset $I \subseteq \{1, \dots, p-1\}$ of indices such that the remainder of $\sum_{i \in I} a_i$ when divided by p is r?

2. Let $f : [0,1] \to [0,1]$ be continuous function non-decreasing on $[0,1]$ and differentiable on $(0,1)$ such that $f'(x) < 1$ for all $x \in (0,1)$. Denote $A_0 = [0,1]$ and $A_{n+1} = f(A_n)$ for all $n = 0, 1, \dots$. Show that $\sup_{x,y \in A_n} |x - y| \to 0$.

3. Let n and m be positive integers and let $A_1, \dots, A_m$ be idempotent $(n \times n)$-matrices (i.e. $A^2 = A$). Let I denote the identity. Prove that

$$\sum_{i=1}^{m} \dim \ker A_i \geq \operatorname{rank}(I - A_1 A_2 \dots A_m).$$

(Here dim denotes the dimension of a vector space; $\ker A = A^{-1}(0)$ is a kernel of a matrix; rank is the rank of a matrix.)

4. Some language uses n letters. Every its word is finite and no word is a prefix of any other word. Let a_k denote the number of words of length $k \in \mathbb{N}$. Prove that

$$\sum_{k \geq 1} \frac{a_k}{n^k} \leq 1.$$

5. Let $x > 0$. Prove that

$$\lim_{n \to \infty} \frac{1^{x-1} + 2^{x-1} + \dots + n^{x-1}}{n^x} = \frac{1}{x}.$$

Solutions

1. Clearly (e.g., by induction), it is enough to show that given $k < p-1$ different nonzero elements $r_1, \ldots, r_k$ of $\mathbb{Z}_p$ and some other nonzero $n \in \mathbb{Z}_p$, one of $r_i + n$ or n itself would differ from all the r_i's.

   Try n, if n is among r_i's, try $2n$ and so on. Since all $n, 2n, \ldots, (p-1)n$ are different, this can only continue until kn. Then $(k+1)n$ will fit the requirements.

2. Since f is non-decreasing, the sequence $(f^n(0))$ is bounded and non-decreasing, while $(f^n(1))$ is bounded and non-increasing. Moreover, each consists of (lower or, respectively, upper) bounds for the other. Take $m := \lim f^n(0)$ and $M := \lim f^n(1)$. Clearly $m \le M$. Since f is continuous they are both fixed points: indeed, e.g., $f(m) = \lim f(f^n(0)) = \lim f^{n+1}(0) = m$. If $M \neq m$, then the mean value theorem gives $\xi \in (m, M)$ such that

$$M - m = f(M) - f(m) = f'(\xi)(M-m) < M - m.$$

   So, $\sup\limits_{x,y \in A_n} |x-y| = f^n(1) - f^n(0) \to 0$.

3. If A is idempotent, then $I - A$ is idempotent and the vector space $V := \mathbb{R}^n$ can be decomposed as $V = A(V) + (I-A)(V)$ with $A(V) \cap (I-A)(V) = \{0\}$. In particular $\operatorname{rank} A = \dim A(V) = n - \dim(I-A)(V) = n - \operatorname{rank}(I-A)$.

   So

$$\operatorname{rank}(I - A_1) = n - \operatorname{rank} A_1 = n - (n - \dim\ker A_1) = \dim\ker A_1.$$

   By induction (using $\operatorname{rank}(A+B) \le \operatorname{rank} A + \operatorname{rank} B$ and $\operatorname{rank}(AB) \le \operatorname{rank} A$), we get that

$$\begin{aligned}\sum_{i<n} \dim\ker A_i + \dim\ker A_n &\ge \operatorname{rank}(I - \prod_{i<n} A_i) + \operatorname{rank}(I - A_n) \\ &\ge \operatorname{rank}((I - \prod_{i<n} A_i)A_n + I - A_n) \\ &= \operatorname{rank}(I - \prod_{i\le n} A_i).\end{aligned}$$

4. Consider first all possible words (perhaps not in the language) of length m: their number is n^m. Given a language word of length $k \le m$, we can have exactly n^{m-k} possible words of length m containing the given word as a prefix. By the assumption none of these possible words contains any other language word as a prefix. Thus, we have

$$\sum_{1\le k\le m} a_k n^{m-k} \le n^m.$$

   Dividing by n^m, we obtain

$$\sum_{1\le k\le m} a_k n^{-k} \le 1,$$

   from where the claim follows after taking the limit on m.

5. One way to do it is to take $f(t) = t^{x-1}$ in the general result below.

   Let $f : [0,1] \to \mathbb{R}$ be monotone on $(0,1)$ such that the (maybe improper) integral $\int_0^1 f(t)dt$ exists. Then

$$\lim_{n\to\infty} \frac{f(\frac{1}{n}) + f(\frac{2}{n}) + \cdots + f(\frac{n-1}{n})}{n} = \int_0^1 f(t)dt.$$

   Indeed, assume, e.g., that f is non-decreasing. Then

$$\int_0^{1-\frac{1}{n}} f(t)dt \le \frac{f(\frac{1}{n}) + f(\frac{2}{n}) + \cdots + f(\frac{n-1}{n})}{n} \le \int_{\frac{1}{n}}^1 f(t)dt.$$