Matemaatika-informaatikateaduskonna üliõpilaste matemaatikaolümpiaad

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1) Tõesta, et

$$\lim_{t\to 0+} \left(\sum_{n=1}^{\infty} \frac{e^{-n^2 t}}{n} \right) / \ln t = -\frac{1}{2}.$$

2) Olgu $P(x)$ trigonomeetriline polünoom astmega n, s.t. $P(x) = \sum_{k=0}^{n} a_k \sin(kx + \varphi_k)$. Tõesta võrratus

$$|P(x)| \le \frac{1}{\pi} \int_0^{2\pi} |P(t)|^2 \, dt + \frac{2n+1}{8}.$$

3) Olgu P polünoom, kusjuures $\deg P \le 2n$ ja $|P(k)| \le 1$ iga $k \in \mathbb{Z} \cap [-n, n]$ korral. Tõesta, et iga $x \in [-n, n]$ korral $|P(x)| \le 2^{2n}$.

4) Tähistagu M_n kõikide kompleksete $n \times n$-maatriksite hulka. Olgu $A \in M_n$ regulaarne. Näita, et järgmised tingimused on samaväärsed:

   (a) A ja $-A$ on sarnased.

   (b) Leiduvad maatriksid B, $C \in M_n$ nii, et $A = B + C$ ja $B^2 + C^2 = 0$.

5) Tõesta võrratus

$$\int_0^1 \sqrt{1 + x^2 \ln^2(1+x)} dx \ge \sqrt{\frac{17}{16}}.$$

6) Olgu $F : \mathbb{R}^2 \to \mathbb{R}$ pidev funktsioon, kusjuures kõikide u, v, w korral

$$F(u, 0) = u, \; F(u, u) = 0, F(F(u, w), F(v, w)) = F(u, v).$$

Tõesta, et leidub rangelt monotoonne funktsioon $f : \mathbb{R} \to \mathbb{R}$ nii, et

$$f(x - y) = F(f(x), f(y))$$

kõikide $x, y \in \mathbb{R}$ korral.

Solutions

1) Since the function $\frac{e^{-n^2t}}{n}$ with argument t is nonnegative and decreasing, we can conclude (as in the proof of integral test for convergence) that

$$0 \le \sum_{n=1}^{\infty} \frac{e^{-n^2t}}{n} - \int_1^{\infty} \frac{e^{-x^2t}}{x} dx \le e^{-t} \le 1.$$

This implies that

$$\lim_{t\to 0+} \left(\sum_{n=1}^{\infty} \frac{e^{-n^2t}}{n} \right) / \ln t = \lim_{t\to 0+} \left(\int_1^{\infty} \frac{e^{-x^2t}}{x} dx \right) / \ln t.$$

Compute

$$\int_1^{\infty} \frac{e^{-x^2t}}{x} dx = \int_{\sqrt{t}}^{\infty} \frac{e^{-y^2}}{y} dy = \int_{\sqrt{t}}^{1} \frac{dx}{x} + \int_{\sqrt{t}}^{1} \frac{e^{-x^2} - 1}{x} dx + \int_1^{\infty} \frac{e^{-x^2}}{x} dx$$

and observe that the latter two integrals have finite limits when $t \to 0+$. Therefore

$$\lim_{t\to 0+} \left(\sum_{n=1}^{\infty} \frac{e^{-n^2t}}{n} \right) / \ln t = \lim_{t\to 0+} \left(\int_{\sqrt{t}}^{1} \frac{dx}{x} \right) / \ln t = \lim_{t\to 0+} \frac{-\ln \sqrt{t}}{\ln t} = -\frac{1}{2}.$$

2) Observe that

$$\frac{1}{\pi} \int_0^{2\pi} |P(t)|^2 \, dt = 2a_0^2 + a_1^2 + \cdots + a_n^2 \ge$$

$$\ge \left(|a_0| - \frac{1}{8} \right) + \left(|a_1| - \frac{1}{4} \right) + \cdots + \left(|a_n| - \frac{1}{4} \right) =$$

$$= |a_0| + |a_1| + \cdots + |a_n| - \frac{2n+1}{8} \ge$$

$$\sum_{k=0}^{n} |a_k \sin(kx + \phi_k)| - \frac{2n+1}{8} \ge |P(x)| - \frac{2n+1}{8}$$

for any $x \in \mathbb{R}$. (We used inequalities

$$2u^2 - u + \frac{1}{8} = 2(u - \frac{1}{4})^2 \ge 0, \; u^2 - u + \frac{1}{4} = (u - \frac{1}{2})^2 \ge 0.)$$

3) According to Lagrange formula we have

$$P(x) = \sum_{k=-n}^{n} P(k) \prod_{i \neq k} \frac{x-i}{k-i}.$$

Since $|P(k)| \leq 1$ for $k \in [-n, n] \cap \mathbb{Z}$ we get

$$|P(x)| \leq \sum_{k=-n}^{n} |P(k)| \prod_{i \neq k} \frac{|x-i|}{|k-i|} \leq \sum_{k=-n}^{n} \prod_{i \neq k} \frac{|x-i|}{|k-i|}.$$

For every $x \in [-n, n]$ the inequality

$$\prod_{i \neq k} |x-i| \leq (2n)!$$

holds. Indeed, in the case $x \geq k$ one gets

$$\prod_{i \neq k} |x-i| = \left(\prod_{i>k} |x-i| \right) \left(\prod_{i<k} |x-i| \right) \leq (n-k)!((n-k+1)\dots 2n) = (2n)!.$$

The case $x < k$ is done similarly. Hence

$$\prod_{i \neq k} \frac{|x-i|}{|k-i|} \leq (2n)! \prod_{i \neq k} \frac{1}{|k-i|} \leq (2n)! \frac{1}{(k+n)!(n-k)!},$$

so

$$|P(x)| \leq \sum_{k=-n}^{n} \frac{(2n)!}{(k+n)!(n-k)!} = \sum_{k=0}^{2n} \frac{(2n)!}{k!(2n-k)!} = \sum_{k=0}^{2n} C_{2n}^{k} = 2^{2n},$$

as needed.

4) (1) $\Rightarrow$ (2). If A is a block matrix of the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where A_1, A_2 are square matrices similar to $-A_1, -A_2$, respectively, and condition (2) is satisfied for A_1, A_2, then it is clearly satisfied for A. We may assume that A is in Jordan normal form. The condition (1) means that for any $k \in \mathbb{N}$, $\alpha \in \mathbb{C}$ the number of Jordan cells of size k corresponding to eigenvalues α and $-\alpha$ is the same. A is non-degenerate, so A does not contain nilpotent Jordan cells. Thus we reduce to the case $A = \begin{pmatrix} \alpha E + J & 0 \\ 0 & -\alpha E + J \end{pmatrix}$.

Here $\alpha \in \mathbb{C} \setminus \{0\}$, E is the identity matrix, J is the nilpotent Jordan cell of the same size $m = \frac{n}{2}$. One obviously can choose a basis $u_1, \dots, u_m, v_1, \dots, v_m$ of $\mathbb{C}^{2n}$ such that

$$Au_i = \alpha u_i + u_{i+1},\ Av_i = \alpha v_i + v_{i+1},\ i = \overline{1, m-1},\ Au_m = \alpha u_m,\ Av_m = -\alpha v_m.$$

Put $x_i = u_i + v_i$, $y_i = u_i - v_i$. One immediately obtains

$Ax_i = \alpha y_i + x_{i+1}$, $Ay_i = \alpha x_i + y_{i+1}$, $i = \overline{1, m-1}$, $Ax_m = \alpha y_m, Ay_m = \alpha x_m$.

Let us define $B, C \in M_n$ by the following equations

$$Bx_i = Ax_i,\ By_i = 0,\ Cx_i = 0,\ Cy_i = Ay_i,\ \text{for odd } i,$$

$$Bx_i = 0,\ By_i = Ay_i,\ Cx_i = Ax_i,\ Cy_i = 0,\ \text{for even } i.$$

It is checked directly that $A = B + C$ and $B^2 = C^2 = 0$.

(2) $\Rightarrow$ (1). Put $V = \mathbb{C}^n$, $V_1 = \mathrm{im}(B)$, $V_2 = \mathrm{im}(C)$. There is a linear map $\phi : V_1 \oplus V_2 \mapsto V$, $(v_1, v_2) \mapsto v_1 + v_2$. The image of ϕ clearly contains $\mathrm{im}(A)$ and, since A is non-degenerate, coincides with V. From the other hand, $\mathrm{im}(B) \subset \ker(B)$, and so $\dim V_1 \leq \dim \frac{V}{2}$. Analogously, $\dim V_2 \leq \dim \frac{V}{2}$. We conclude, that $V = V_1 \oplus V_2$. It follows directly from the construction that $B(V_1) = 0$, $C(V_2) = 0$, $B(V_2) = V$, $C(V_1) = V_2$. Denote by I the linear transformation of V such that $I|_{V_1} = \mathrm{id}$, $I|_{V_2} = -\mathrm{id}$. One checks directly that $IB = -B$, $BI = B$, $IC = C$, $CI = -C$. Thus $IAI^{-1} = -A$.

5) Given integral can be viewed as the length of a curve defined by the function $F(x) = \int x \ln(1+x)dx$ on interval $[0, 1]$. After computing

$$F(x) = \int x \ln(1+x)dx = \frac{1}{4}(2(x^2-1)\ln(x+1) - (x-2)x)$$

notice that F is monotone, $F(0) = 0$ and $F(1) = \frac{1}{4}$. Hence, the length of the curve defined by F is not less than the distance between $(0, 0)$ and $(1, \frac{1}{4})$, which is $\sqrt{\frac{1}{4^2} + 1^2} = \sqrt{\frac{17}{16}}$.

6) Define a function $\varphi(x) \equiv F(0, x)$. It is obviously continuous. We have

$$\varphi(\varphi(x)) = F(0, F(0, x)) = F(F(x, x), F(0, x)) = F(x, 0) = x \qquad (1)$$

for all $x \in \mathbb{R}$, hence φ is bijective. Therefore φ is strictly monotone. Also check that $\varphi(F(x, y)) \equiv F(y, x)$.

Consider equation $y = F(t, x)$ and try to find solutions to it. It is easy to observe that

$$y = F(t, x) \Longleftrightarrow t = F(y, \varphi(x)) \Longleftrightarrow x = F(\varphi(y), \varphi(t)). \qquad (2)$$

Now for every $y \in \mathbb{R}$ consider functions $F^y(x) \equiv F(y, x)$ and $F_y(x) \equiv F(x, y)$. Using (2) it is easy to check that these functions are bijective. Hence, strictly monotone.

Assume φ is increasing. Then it is easy to check that $\varphi(x) \equiv x$, which means $F(x,y) \equiv F(y,x)$ and $F_y = F^y$. Take $x_0 > y_0 > 0$. Then $F_{y_0}(0) = y_0 > 0 = F_{y_0}(y_0)$, which means F_{y_0} is decreasing. Hence $F(x_0,y_0) = F_{y_0}(x_0) < 0$. Likewise F_{x_0} is decreasing and $F(x_0,y_0) = F_{x_0}(y_0) > F_{x_0}(x_0) = 0$, which is a contradiction.

Thus φ is decreasing. Note that $\varphi(0) = 0$. Now as $F_y(y) = 0 > \varphi(y) = F_y(0)$ iff $y > 0$ we see that the functions F_y $(y \in \mathbb{R})$ are increasing. Similarly the functions F^y $(y \in \mathbb{R})$ are decreasing.

If function f satisfying

$$f(x-y) \equiv F(f(x), f(y)), \tag{3}$$

exists then due to (2) we get

$$f(x) \equiv F(f(x-y), \varphi(f(y)))$$

and particularly

$$f(2x) \equiv F(f(x), \varphi(f(x))).$$

More generally, for arbitrary constant $n \in \mathbb{N}$ holds

$$f(nx) \equiv F(f(x), \varphi(f((n-1)x))).$$

This gives us an idea to define *"multiplying"* functions as following:

$$\psi_0(x) \equiv 0,$$

$$\psi_n(x) \equiv F(x, \varphi(\psi_{n-1}(x)))$$

for each $n \in \mathbb{Z}^+$. Next we define $\psi_{-n} := \varphi\psi_n$ for $n \in \mathbb{Z}^+$ or equivalently

$$\psi_{-n}(x) \equiv \varphi(\psi_n(x)).$$

Applying (1) to the latter when necessary we get the same equality $\psi_{-n} = \varphi\psi_n$ for every $n \in \mathbb{Z}$. Observe that for a function f satisfying (3) then holds

$$\psi_n(f(x)) \equiv f(nx).$$

Now we show that for all $n, m \in \mathbb{Z}$ holds

$$\psi_{n+m}(x) \equiv F(\psi_m(x), \psi_{-n}(x)). \tag{4}$$

Assume first $n, m \geq 0$. We show (4) with induction by m. In cases $m = 0$ and $m = 1$ the assertion is true. Assuming that (4) holds for $m = \mu \geq 1$ we show that it also holds for $m = \mu + 1$:

$$\psi_{n+\mu+1}(x) \equiv F(\psi_\mu(x), \psi_{-n-1}(x)),$$

whereas

$$\psi_{-n-1}(x) \equiv \varphi(\psi_{n+1}(x)) \equiv \varphi(F(x, \psi_{-n})) \equiv F(\psi_{-n}, x)$$

and induction hypothesis $\psi_{\mu+1}(x) \equiv F(\psi_\mu(x), \psi_{-1}(x))$ implies with respect to (2)

$$\psi_\mu(x) \equiv F(\psi_{\mu+1}(x), x).$$

Therefore, indeed

$$\psi_{n+\mu+1}(x) \equiv F(F(\psi_{\mu+1}(x), x), F(\psi_{-n}(x), x)) \equiv F(\psi_{\mu+1}(x), \psi_{-n}(x)).$$

Now consider $n \geq 0$, $-n < m < 0$. Then $-m > 0$ and $m + n < 0$. Therefore holds $\psi_n(x) \equiv F(\psi_{-m}(x), \psi_{-m-n}(x))$, which gives $\psi_{m+n}(x) \equiv F(\psi_m(x), \psi_{-n}(x))$. If $n \geq 0$, $m < -n$ we have $m+n < 0$ and thus $\psi_{-m}(x) \equiv F(\psi_n(x), \psi_{m+n}(x))$, which gives $\psi_{m+n}(x) \equiv F(\psi_m(x), \psi_{-n}(x))$.

Equation (4) means that ψ as a function of its subscript satisfies (3) for integer arguments, and is therefore *multiplying* for itself. Hence $\psi_n\psi_m \equiv \psi_{nm}$ for all $n, m \in \mathbb{Z}$. Also observe that by definition ψ_n is bijective whenever $n \neq 0$. This allows us to define $\psi_{n^{-1}} := \psi_n^{-1}$ for $n \neq 0$. It is now clear that we can define ψ_q for any $q \in \mathbb{Q}$ with all the same properties.

Denote $f_x(q) = \psi_q(x)$ for $x \in \mathbb{R}$, $q \in \mathbb{Q}$. Using the fact that functions φ and F^y are strictly monotone it is easy to check that f_x is increasing if $x > 0$. Fix such an x and let q, q_i denote exclusively elements of $\mathbb{Q}$.

Next, define $f_x(r) = \inf_{q>r} f_x(q)$ for $r \in \mathbb{R}\backslash\mathbb{Q}$. Then $f_x(q_1) < f_x(r) < f_x(q_2)$ for all $q_1 < r < q_2$. Indeed, e.g. if there was $q_0 > r$ such that $q_0 = \inf_{q>r} f_x(q)$ then the same equality would hold for any $q \in (r, q_0) \cap \mathbb{Q}$, a contradiction since f_x is strictly monotone on $\mathbb{Q}$. Also observe that $f_x(r) = \inf_{q \geq r} f_x(q)$ holds for any $r \in \mathbb{R}$.

We only need to check condition (3) now. Take $r_1, r_2 \in \mathbb{R}$. Then

$$F(f_x(r_1), f_x(r_2)) = F(\inf_{q_1 \geq r_1} f_x(q_1), \inf_{q_2 \geq r_2} f_x(q_2)) = \inf_{q_1 \geq r_1} \sup_{q_2 \geq r_2} F(f_x(q_1), f_x(q_2))$$

$$= \inf_{q_1 \geq r_1} \sup_{q_2 \geq r_2} f_x(q_1 - q_2) = \inf_{q_1 \geq r_1} \sup_{q_2 \geq r_2} \varphi(f_x(q_2 - q_1)) = \inf_{q_1 \geq r_1} \varphi(\inf_{q_2 \geq r_2} f_x(q_2 - q_1))$$

$$= \inf_{q_1 \geq r_1} \varphi(f_x(r_2 - q_1)) = \inf_{q_1 \geq r_1} f_x(q_1 - r_2) = f_x(r_1 - r_2).$$

Hereby we used the facts that functions φ, F_y and F^y are continuous and strictly monotone, as well as equality $f_x(-r) = \varphi(f_x(r))$.