Matemaatika võistlus

Tartu, 11.10.2019

1. Olgu A, B ja C reaalväärtustega maatriksid suurusega 5×5, kusjuures

$$\left(B + \left(A - \left(A^{-1} + \left(B^{-1} - A\right)^{-1}\right)^{-1}\right)^{-1}\right) A = C + C^{-1}$$

ning kõik ülalolevas avaldises esinevad pöördmaatriksid eksisteerivad. Leia vähemalt üks sobiv maatriks C ja avalda see maatriksite A ja B kaudu.

2. Tõesta, et kõikide positiivsete täisarvude n korral

$$\left(\frac{7+\sqrt{61}}{2}\right)^n + \left(\frac{7-\sqrt{61}}{2}\right)^n$$

on täisarv. Milliste n korral see jagub 7-ga?

3. Olgu funktsioon $f : \mathbb{R} \to \mathbb{R}$ diferentseeruv. Kas alati leidub $x \in [0, 1]$ nii, et

$$4\left(f(1) - f(0)\right)^2 + \left(\pi x f'(x)\right)^2 = \pi^2 \cdot f'(x)^2?$$

4. Olgu A ja B sama järku reaalväärtustega ruutmaatriksid, kusjuures

$$AB - BA = 2A.$$

   (a) Tõesta, et $A^k B - BA^k = 2kA^k$ iga $k \in \mathbb{N}$ korral.

   (b) Tõesta, et leidub $k \in \mathbb{N}$ nii, et $A^k = 0$.

5. (a) Tõesta, et

$$\frac{\sin n}{n}\left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right) \xrightarrow[n\to\infty]{} 0.$$

   (b) Kas

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}\left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right)$$

   koondub?

Math Competition

Tartu, 11.10.2019

1. Let A, B, and C be 5×5 realvalued matrices such that

$$\left(B + \left(A - \left(A^{-1} + \left(B^{-1} - A\right)^{-1}\right)^{-1}\right)^{-1}\right) A = C + C^{-1}$$

   and such that all inverses appearing above exist. Find at least one such matrix C and represent it as formula depending on A and B.

2. Prove that

$$\left(\frac{7+\sqrt{61}}{2}\right)^n + \left(\frac{7-\sqrt{61}}{2}\right)^n$$

   is an integer for all positive integers n. For which n is this number divisible by 7?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Does there always exist $x \in [0, 1]$ such that

$$4\left(f(1) - f(0)\right)^2 + (\pi x f'(x))^2 = \pi^2 \cdot f'(x)^2?$$

4. Let A and B be square realvalued matrices of the same size such that

$$AB - BA = 2A.$$

   (a) Prove that $A^k B - BA^k = 2kA^k$ for all $k \in \mathbb{N}$.

   (b) Prove that there exists $k \in \mathbb{N}$ such that $A^k = 0$.

5. (a) Prove that

$$\frac{\sin n}{n}\left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right) \xrightarrow[n\to\infty]{} 0.$$

   (b) Does the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}\left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right)$$

   converge?

Vihjed

1. Vt millega võrdub nende kolme suure pöördmaatriksi (ehk vastavate maatriksite) korrutis.
2. Need liidetavad on mingi polünoomi juured! Kas tuleb rekurrentne seos?
3. Lagrange'i kvt rakendatud kahe sobiva funktsiooni vahele.
4. (a) Lihtne.
   (b) Uuri operaatori $L : A \mapsto AB - BA$ omaväärtused.
5. (a) Paremat kordajat saab hinnata integraaltestiga.
   (b) $\sum \sin(n)$ on tõkestatud (kui mõelda e astmetega). Ülejäänud kordaja monotoonselt läheb nulli.

1. Denote $G_1 = B^{-1} - A$, $G_2 = A^{-1} + G_1^{-1}$, and $G_3 = A - G_2^{-1}$. Note that

$$G_3 \cdot G_2 \cdot G_1 = \left(I + AG_1^{-1} - I\right) \cdot G_1 = A.$$

So the left hand side of the original equation is equal to

$$(B + G_2 \cdot G_1 \cdot A^{-1})A = BA + G_2 \cdot G_1,$$

where

$$G_2 \cdot G_1 = A^{-1}G_1 + I = A^{-1}B^{-1} - I + I = (BA)^{-1}.$$

So we can take $C = BA$.

2. Note that $x_i = \dfrac{7 \pm \sqrt{61}}{2}$ are the roots of $x^2 - 7x - 3 = 0$. So $x_i^{n+2} - 7x_i^{n+1} - 3x_i^n = 0$ for all $n = 0, 1, 2, \ldots$. Hence also

$$x_1^{n+2} + x_2^{n+2} = 7(x_1^{n+1} + x_2^{n+1}) + 3(x_1^n + x_2^n)$$

for all $n = 0, 1, 2, \ldots$. Since $x_1 + x_2 = 7$, it is clear that the claim follows. Moreover, 7 divides $x_1^n + x_2^n$ if and only if n is odd.

3. If $f(1) = f(0)$ then take $x = 1$. Otherwise assume $x \neq 1$ and write

$$\frac{4}{\pi^2(1 - x^2)} = \frac{f'(x)^2}{(f(1) - f(0))^2}.$$

So it is enough to find x such that

$$\frac{2}{\pi\sqrt{1 - x^2}} = \frac{f'(x)}{f(1) - f(0)},$$

which is the same as

$$\left(\frac{2 \arcsin x}{\pi} - \frac{f(x)}{f(1) - f(0)}\right)' = 0.$$

Denote the left hand side by $F'(x)$ and note that the claim follows from Rolle's (or mean value) theorem, because $F(1) = F(0)$. Indeed, F is continuous on $[0, 1]$, differentiable on $(0, 1)$, and

$$F(1) - F(0) = \frac{2}{\pi}(\arcsin(1) - \arcsin(0)) - \frac{f(1) - f(0)}{f(1) - f(0)} = 1 - 1 = 0.$$

4. (a)

$$A^{n+1}B - BA^{n+1} = A(A^nB) - A(BA^n) + (AB)A^n - (BA)A^n = A(2nA^n) + (2A)A^n = 2(n+1)A^{n+1}.$$

(b) *Solution 1.* Consider a linear operator $L : \mathrm{Mat}_n \to \mathrm{Mat}_n$ defined by $X \mapsto XB - BX$. By the above, $2k$ is its eigenvalue whenever $A^k \neq 0$. But it has at most n^2 eigenvalues, so A^k must be 0 for some k.

Solution 2 (found by Urve). Consider matrix norms in (a) (recall that the matrix norm satisfies the triangle inequality and a similar inequality for products of matrices):

$$2k\|A^k\| = \|2kA^k\| = \|A^kB - BA^k\| \leq \|A^kB\| + \|BA^k\| \leq 2\|A^k\|\|B\|$$

for all $k \in \mathbb{N}$. So either claim holds or $\|B\| \geq k$ for all $k \in \mathbb{N}$, which is impossible.

5. (a) *Solution 1.* The proof of the integral test gives

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 1 + \int_1^n \frac{1}{\sqrt{x}} dx = 1 + \left(2\sqrt{x}\Big|_1^n\right) = 2\sqrt{n} - 1,$$

so that

$$\frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right) \leq \frac{2}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0,$$

which implies the claim, because $\sin n$ is bounded.

Solution 2 (found by Natalia). Let us estimate using the over-harmonic series:

$$\left|\frac{\sin n}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right| \leq \frac{1}{n^{1/4}} \sum_{k=1}^{n} \frac{1}{k^{5/4}} \leq \frac{1}{n^{1/4}} \zeta\left(\frac{5}{4}\right) \xrightarrow[n\to\infty]{} 0,$$

where $\zeta\left(\frac{5}{4}\right) = \sum_{k=1}^{\infty} \frac{1}{k^{5/4}} < \infty$.

(b) The above convergence is actually monotone. Indeed,

$$\frac{1}{n+1}\left(\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}}\right) < \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right) \iff n\left(\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}}\right) < (n+1)\left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right) \iff \frac{n}{\sqrt{n+1}} < \sum_{k=1}^{n} \frac{1}{\sqrt{k}},$$

which is clear. On the other hand,

$$\left|\sum_{n=1}^{N} \sin n\right| = \left|\sum_{n=1}^{N} \frac{e^{in} - e^{-in}}{2}\right| = \frac{1}{2}\left|\frac{e^{i(N+1)} - 1}{e^i - 1} - \frac{e^{-i(N+1)} - 1}{e^{-i} - 1}\right| \leq \frac{1}{2}\left(\frac{|e^{i(N+1)}| + 1}{|e^i - 1|} + \frac{|e^{-i(N+1)}| + 1}{|e^{-i} - 1|}\right)$$
$$\leq \frac{1}{2}\left(\frac{2}{m} + \frac{2}{m}\right) = \frac{2}{m},$$

where $m = |e^i - 1| = |e^{-i} - 1| > 0$ (in fact, $m = \sqrt{(\cos(1) - 1)^2 + \sin^2(1)} = \sqrt{2 - 2\cos(1)}$).

By the way, there is another way to see that these partial sums are bounded, using trigonometry instead of the complex representation (*found by Natalia*):

$$\sum_{n=1}^{N} \sin n = \frac{1}{2\sin(\frac{1}{2})} \sum_{n=1}^{N} 2\sin(\tfrac{1}{2}) \sin n = \frac{1}{2\sin(\frac{1}{2})} \sum_{n=1}^{N} \left[\cos\left(\frac{2n-1}{2}\right) - \cos\left(\frac{2n+1}{2}\right)\right]$$
$$= \frac{\cos(\frac{1}{2}) - \cos(\frac{1+2N}{2})}{2\sin(\frac{1}{2})}.$$

The convergence of the original series now follows from Dirichlet's test: if (a_n) and (b_n) are sequences such that (a_n) monotonely goes to 0 and $|\sum_{n=1}^{N} b_n|$ is bounded (by, e.g., $M > 0$), then

$$\sum_{n=1}^{\infty} b_n \cdot a_n$$

converges.

Dirichlet's test can be proven relatively easily using summation by parts:

$$\sum_{n=1}^{N} b_n \cdot a_n = \sum_{n=1}^{N} \left(\sum_{k=1}^{n} b_k\right) \cdot a_n - \sum_{n=2}^{N} \left(\sum_{k=1}^{n-1} b_k\right) a_n = a_{N+1} \cdot \left(\sum_{k=1}^{N} b_k\right) + \sum_{n=1}^{N} \left(\sum_{k=1}^{n} b_k\right) \cdot (a_n - a_{n+1}).$$

Here the first summand goes to 0 and the second converges absolutely:

$$\sum_{n=1}^{N} \left| \sum_{k=1}^{n} b_k \right| \cdot |a_n - a_{n+1}| \leq M \sum_{n=1}^{N} (a_n - a_{n+1}) = M(a_1 - a_{N+1}) \to M a_1 .$$