The **differential equation** of damped oscillations: $x'' + 2\beta x' + \omega_0^2 x = 0$

Its solution in the complex form (the complex deviation) is: $x = A(t)\, e^{i\omega t}$. The real deviation or coordinate $x = A \cos \omega t$ is the real part of the complex deviation.

We need to find the dependence of the amplitude $A(t)$ on time and the angular frequency ω.

So we take the first derivative according to the rule of differentiation of the product of two functions, which in our case are $A(t)$ and $e^{i\omega t}$:

$x' = dx/dt = A'$ (derivative of the 1-st function) x $e^{i\omega t}$ (the 2-nd function) +

+ A (the 1-st function) x $e^{i\omega t}$ (derivative of the outer function) x $(i\omega)$ (derivative of the inner function).

The second derivative: $x'' = A'' e^{i\omega t} + A' e^{i\omega t}(i\omega) + A' e^{i\omega t}(i\omega) + A\, e^{i\omega t}(i\omega)(i\omega) =$

$= A'' e^{i\omega t} + 2\, i\omega A' e^{i\omega t} - \omega^2 A\, e^{i\omega t}$

Now we substitute the derivatives into the initial equation:

$$A'' e^{i\omega t} + 2\, i\omega A' e^{i\omega t} - \omega^2 A\, e^{i\omega t} + 2\beta A' e^{i\omega t} + 2\, i\omega \beta A\, e^{i\omega t} + \omega_0^2 A\, e^{i\omega t} = 0$$

$$A'' + 2\, i\omega A' - \omega^2 A + 2\beta A' + 2\beta A\, i\omega + \omega_0^2 A = 0$$

Terms from $d^2x/dt^2 = x''$ — terms from $dx/dt = x'$ — terms from $x(t)$

and we get the equation $A'' + 2\beta A' + (\omega_0^2 - \omega^2) A = 0$ for the real part

and $2\, i\omega A' + 2\beta A\, i\omega = 2\, i\omega (A' + \beta A) = 0$ for the imaginary part

The trivial solution is: $\omega = 0$ and the non-trivial one is: $\mathbf{A' + \beta A = 0}$

We get $A' = -\beta A$ or $dA/dt = -\beta A$ or $dA/A = -\beta\, dt$.

Integrating this gives us the equation: $\ln A = -\beta t + \text{const}$

The initial condition: $\text{const} = \ln A_0$ where $A_0 = A\,(t = 0)$

Alltogether we have: $\ln A - \ln A_0 = -\beta t$ or $\ln (A/A_0) = -\beta t$

We take the antilogarithm and so we get : $A/A_0 = e^{-\beta t}$ or $\mathbf{A = A_0\, e^{-\beta t}}$

$A' = -\beta A_0\, e^{-\beta t} = -\beta A$

$A'' = (-\beta)(-\beta) A_0\, e^{-\beta t} = \beta^2 A$

We substitute these derivatives into the equation for the real part:

$A'' + 2\beta A' + (\omega_0^2 - \omega^2) A = 0$

and so we get $\beta^2 A + 2\beta(-\beta A) + (\omega_0^2 - \omega^2) A = 0,$

or dividing by A : $\beta^2 - 2\beta^2 + \omega_0^2 - \omega^2 = 0,$

or $-\beta^2 + \omega_0^2 - \omega^2 = 0$ or $\mathbf{\omega^2 = \omega_0^2 - \beta^2}$